MAT 200–Calculus and Analytic Geometry II

Fall 2018

Prerequisites: MAT 150 or MAT 180 or SEM 1101

General Information

Class Schedule: Tuesdays and Thursdays 9:00–10:50am
Classroom: Plato (SR 3D)
Instructor: Michael Daniel Samson
Contact: `mat200@mdvsamson.work`, +65 6577 1944
Class Webpage: Moodle, `mdvsamson.work/mat200`
Office Hours: Tuesdays and Thursdays 11:00–12:00pm, Mondays 2:00–7:00pm or by appointment (through email)

Description

This course builds on the introduction to calculus in MAT 150. Topics in integration include applications of the integral in physics and geometry, and techniques of integration. The course also covers sequences and series of real numbers, power series and Taylor series, and calculus of transcendental functions. Further topics may include a basic introduction to concepts in multivariable and vector calculus.

Course Objectives and Learning Outcomes

Upon completing this course students should be able to:

- Understand the concept of definite integral as a limit of Riemann sums.
- Find indefinite and improper integrals using different integration techniques.
- Perform standard operations with convergent power series, find Taylor and Maclaurin representations.
- Use integrals to solve applied problems and analyze graphs of curves.

Textbooks

CALCULUS Early Transcendentals 8e, International Metric Version, James Stewart, Cengage Learning, ISBN-10 1-305-27237-4, ISBN-13 978-1-305-27237-8

Optional Textbooks

CALCULUS Early Vectors, James Stewart, Brooks/Cole Cengage Learning, ISBN-10 0-534-34941-2, ISBN-13 978-0-534-49348-6

Outline and Tentative Dates

The following schedule is subject to change.

Integration and Some Techniques
September 4: Summations
September 6: Fundamental Theorem of Calculus
September 11: Substitution Rule, **quiz**
September 13: Integration by Parts
September 18: Trigonometric Integrals
September 20: Trigonometric Substitution
September 25: Partial Fractions, **quiz**
September 27: ⋆ Improper Integrals
October 2: **Examination** (discussion on October 4)
Approximation of Definite Integrals as Infinite Sums
October 9: Approximation of Integrals
October 11: Sequences
October 16: Series
October 18: Tests of Series Convergence
October 23: Truncation Error, **quiz**
October 25: Power Series
October 30: Taylor and Maclaurin Series, **quiz**
November 1: Applications of Taylor Series
November 6: *Deepavali*
November 8: **Examination** (discussion on November 13)
Geometric and Physical Applications of Integration
November 15: Areas Between Curves,Volumes
November 20: Cylindrical Shells, **quiz**
November 24: Work, ⋆ Average Value of a Function
November 27: Arc Length
November 29: Areas of Surfaces of Revolution
December 4: Applications in Other Fields, **quiz**
December 6: ⋆ Further Applications of Integration
December 10–14: **Examination** (schedule to be announced by DigiPen Admin)

Grading Policy

The examination on week fifteen is *optional*. You must inform the instructor of your decision to not take the final exam *by week fourteen*.

The relative weights of homework, quizzes and exams are:

5% Homework (at least ten)
7.5% Laboratory Worksheets (at least ten, given during the laboratory sessions)
27.5% Quizzes (given during the laboratory session on the noted dates)
60% Examinations (drop the lowest)

Grades will be computed out of 40 points. Letter grades will be computed subject to:

35 = at least A
30 = at least B
20 = at least C- (passing)

To pass the course, you need to

have a passing examination average *and* ***the course total should be greater than or equal to 20****.*

Late Policy

Late assignments **will not** be accepted. There will be **no make-up** quizzes or exams, unless authorized by the instructor.

On Use of Calculators

Calculator use is *discouraged* for this course, and *will not be allowed* during examinations. More sophisticated computing devices now regularly dispense as output the details taught in the course, without the benefit of understanding the result. Calculators can be useful for doing away with the tedious clerical nature of computation, but this course will not evaluate students on their arithmetic—for most part, in-examination computations will be allowed to be left unsimplified without penalty.

Last Day to Withdraw

The final date to withdraw from this course is **28 October 2018**. Scores for six (6) homework submissions, six (6) laboratory worksheets, two (2) quizzes and one (1) examination should be available before this date. In order to withdraw from a course, in accordance with policy, contact your advisor or the Registrar to begin the withdrawal process—it is *not sufficient* simply to stop attending class or to inform the instructor. The last day for withdrawal from this course is cited in the official catalog.

Academic Integrity Policy

Academic dishonesty *in any form* will not be tolerated in this course. Cheating, copying, plagiarizing, or any other form of academic dishonesty (including doing someone else's individual assignments) will result in, at the very minimum, a zero on the assignment in question, and could result in a failing grade in the course or even expulsion from DigiPen.

External Preparation

It is expected that the students in this class spend eight (8) hours on average per week for outside classroom activities through the semester, including, but not limited to, homework, reading assignments, project implementation, group discussions, preparation of examinations, etc.

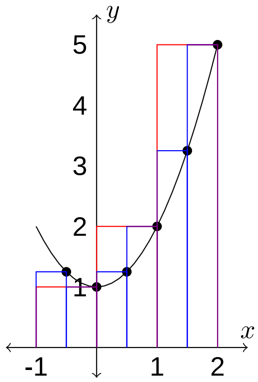
Disability Support Service

Students who have special needs or medical conditions and require formal accommodations in order to fully participate or effectively demonstrate learning in this class should contact the Student Life & Advising Office (`studentlife.sg@digipen.edu`) at the beginning of each semester. A Student Life & Advising Officer will meet with the student privately to discuss how the accommodations will be implemented.

Name: ______________________________

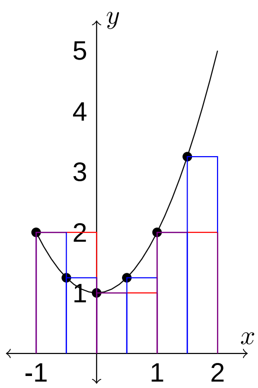
§5.1 #5 (a) Estimate the area under the graph of $f(x) = 1+x^2$ from $x = -1$ to $x = 2$ using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.

For three rectangles, $A \approx 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 5 = 8$. For six rectangles, $A \approx \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{13}{4} + \frac{1}{2} \cdot 5 = \frac{55}{8} = 6.875$.



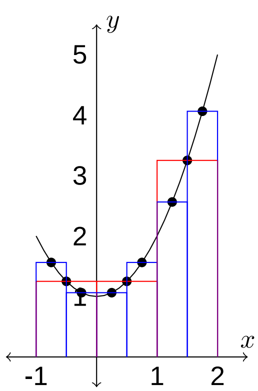
(b) Repeat part (a) using left endpoints.

For three rectangles, $A \approx 1 \cdot 2 + 1 \cdot 1 + 1 \cdot 2 = 5$. For six rectangles, $A \approx \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{13}{4} = \frac{43}{8} = 5.375$.



(c) Repeat part (a) using midpoints.

For three rectangles, $A \approx 1 \cdot \frac{5}{4} + 1 \cdot \frac{5}{4} + 1 \cdot \frac{13}{4} = \frac{23}{4} = 5.75$. For six rectangles, $A \approx \frac{1}{2} \cdot \frac{25}{16} + \frac{1}{2} \cdot \frac{17}{16} + \frac{1}{2} \cdot \frac{17}{16} + \frac{1}{2} \cdot \frac{25}{16} + \frac{1}{2} \cdot \frac{41}{16} + \frac{1}{2} \cdot \frac{65}{16} = \frac{95}{16} = 5.9375$.



(d) From your sketches in parts (a)–(c), which appears to be the best estimate?
As using right endpoints tended to overestimate, and using left endpoints tended to underestimate, the best estimates appear to be those using midpoints. As can be determined, the area is 6 square units.

§5.1 #21 Use

The *area* A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

to find an expression for the area under the graph of $f = \dfrac{2x}{x^2+1}$, $1 \leq x \leq 3$, as a limit. Do not evaluate the limit.

Using n partitions, $\Delta x = \dfrac{2}{n}$, and $x_i = 1 + \dfrac{2i}{n}$. Thus,

$$A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \frac{2+(4i/n)}{[1+(2i/n)]^2+1} = 4 \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n+2i}{(n+2i)^2+n^2}.$$

It can be determined later that $A = \ln 5$.

§5.1 #27 Let A be the area under the graph of an increasing continuous function f from a to b, and let L_n and R_n be the approximations to A with n subintervals using left and right endpoints, respectively.

(a) How are A, L_n and R_n related?
Since f is increasing throughout the interval, the left endpoints are minimums over the subintervals, with the right endpoints are maximums over the subintervals, so $L_n < A < R_n$.

(b) Show that $R_n - L_n = \dfrac{b-a}{n}[f(b) - f(a)]$. Then draw a diagram to illustrate this equation by showing that the n rectangles representing $R_n - L_n$ can be reassembled to form a single rectangle whose area is the right side of the equation.

If the partition points are $a = x_0, x_1, \ldots, x_{n-1}, x_n = b$, with $\Delta x = x_i - x_{i-1} = \dfrac{b-a}{n}$, for $1 \leq i \leq n$, then $R_n = \Delta x[f(x_1) + \cdots + f(x_n)]$ and $L_n = \Delta x[f(x_0) + \cdots + f(x_{n-1})]$, so $R_n - L_n = \Delta x[f(x_n) - f(x_0)] = \dfrac{b-a}{n}[f(b) - f(a)]$.
In the diagram, for each subinterval $[x_{i-1}, x_i]$, the difference between the approximations for the interval is a rectangle whose area is $\Delta x[f(x_i) - f(x_{i-1})]$. If these rectangles are stacked on top of each other, their area would be $\Delta x \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$, which telescopes to $\Delta x[f(x_n) - f(x_0)]$—that is, it forms a rectangle whose width is Δx, and whose length is $f(x_n) - f(x_0)$.

(c) Deduce that $R_n - A < \dfrac{b-a}{n}[f(b) - f(a)]$.

Since $L_n < A < R_n$, $R_n - A < R_n - L_n = \dfrac{b-a}{n}[f(b) - f(a)]$.

§5.2 #67 Which of the integrals $\displaystyle\int_1^2 \arctan x \,\mathrm{d}x$, $\displaystyle\int_1^2 \arctan \sqrt{x} \,\mathrm{d}x$, and $\displaystyle\int_1^2 \arctan(\sin x) \,\mathrm{d}x$ has the largest value? Why?

By Property 7, the order of these integrals can be determined by the order of the integrated functions $\arctan x$, $\arctan \sqrt{x}$ and $\arctan(\sin x)$ on the interval $[1, 2]$. $\arctan x$ is increasing everywhere, so the order of the functions can be determined by the order of the composed functions x, $\sqrt{x}$ and $\sin x$ on the interval $[1, 2]$—there, $\sin x \leq \sqrt{x} \leq x$. Thus, the largest of the values is $\displaystyle\int_1^2 \arctan x \,\mathrm{d}x$.

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

Name: ______________________________

§5.5 #78 Evaluate $\int_0^1 x\sqrt{1-x^4}\,\mathrm{d}x$ by making a substitution and interpreting the resulting integral in terms of an area.

Let $u = x^2$: then $\mathrm{d}u = 2x\,\mathrm{d}x$ and $\int_0^1 x\sqrt{1-x^4}\,\mathrm{d}x = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,\mathrm{d}u$. This is half the area of the portion of the unit circle in the first quadrant, thus $\int_0^1 x\sqrt{1-x^4}\,\mathrm{d}x = \frac{1}{2}\left[\frac{1}{4}\pi\right] = \frac{\pi}{8}$.

§5.5 #94 (a) If f is continuous, prove that $\int_0^{\pi/2} f(\cos x)\,\mathrm{d}x = \int_0^{\pi/2} f(\sin x)\,\mathrm{d}x$.

Using the substitution $u = \frac{\pi}{2} - x$, $x = \frac{\pi}{2} - u$, and $\mathrm{d}u = -\,\mathrm{d}x$, so

$$\int_0^{\pi/2} f(\cos x)\,\mathrm{d}x = \int_{\pi/2}^0 -f\left[\cos\left(\frac{\pi}{2} - u\right)\right]\,\mathrm{d}u = \int_0^{\pi/2} f(\sin u)\,\mathrm{d}u = \int_0^{\pi/2} f(\sin x)\,\mathrm{d}x.$$

(b) Use part (a) to evaluate $\int_0^{\pi/2} \cos^2 x\,\mathrm{d}x$ and $\int_0^{\pi/2} \sin^2 x\,\mathrm{d}x$.

Since

$$\int_0^{\pi/2} \cos^2 x\,\mathrm{d}x + \int_0^{\pi/2} \sin^2 x\,\mathrm{d}x = \int_0^{\pi/2} (\cos^2 x + \sin^2 x)\,\mathrm{d}x = \int_0^{\pi/2} \mathrm{d}x = \frac{\pi}{2},$$

and $f(x) = x^2$ is continuous, from above, $\int_0^{\pi/2} \cos^2 x\,\mathrm{d}x = \int_0^{\pi/2} \sin^2 x\,\mathrm{d}x = \frac{1}{2}\left[\frac{\pi}{2}\right] = \frac{\pi}{4}$.

§7.1 #32 Evaluate $\int_1^2 \frac{(\ln x)^2}{x^3}\,\mathrm{d}x$.

Let $u = (\ln x)^2$: then $\mathrm{d}u = \frac{2\ln x\,\mathrm{d}x}{x}$, $\mathrm{d}v = \frac{\mathrm{d}x}{x^3}$, so $v = -\frac{1}{2x^2}$ and

$$\int_1^2 \frac{(\ln x)^2}{x^3}\,\mathrm{d}x = \left[-\frac{1}{2}\left(\frac{\ln x}{x}\right)^2\right]_1^2 + \int_1^2 \frac{\ln x}{x^3}\,\mathrm{d}x.$$

Let $U = \ln x$: then $\mathrm{d}U = \frac{\mathrm{d}x}{x}$, $\mathrm{d}V = \frac{\mathrm{d}x}{x^3}$, so $V = -\frac{1}{2x^2}$ and

$$\int_1^2 \frac{(\ln x)^2}{x^3}\,\mathrm{d}x = \left[-\frac{(\ln x)^2 + \ln x}{2x^2}\right]_1^2 + \frac{1}{2}\int_1^2 \frac{\mathrm{d}x}{x^3} = \left[-\frac{2(\ln x)^2 + 2\ln x + 1}{4x^2}\right]_1^2$$
$$= \frac{1}{4} - \frac{2(\ln 2)^2 + 2\ln 2 + 1}{16} = \frac{3 - \ln 4 - 2(\ln 2)^2}{16} \approx 0.0408.$$

§7.1 #70 If $f(0) = g(0) = 0$ and f'' and g'' are continuous, show that

$$\int_0^a f(x)g''(x)\,\mathrm{d}x = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,\mathrm{d}x.$$

Let $u = f(x)$: then $\mathrm{d}u = f'(x)\,\mathrm{d}x$, $\mathrm{d}v = g''(x)\,\mathrm{d}x$, so $v = g'(x)$ and

$$\int_0^a f(x)g''(x)\,\mathrm{d}x = [f(x)g'(x)]_0^a - \int_0^a f'(x)g'(x)\,\mathrm{d}x = f(a)g'(a) - \int_0^a f'(x)g'(x)\,\mathrm{d}x.$$

Likewise,

$$\int_0^a f''(x)g(x)\,\mathrm{d}x = [f'(x)g(x)]_0^a - \int_0^a f'(x)g'(x)\,\mathrm{d}x = f'(a)g(a) - \int_0^a f'(x)g'(x)\,\mathrm{d}x,$$

thus $\int_0^a f'(x)g'(x)\,\mathrm{d}x = f'(a)g(a) - \int_0^a f''(x)g(x)\,\mathrm{d}x$, and the statement follows by substitution.

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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§5.5 #77 Evaluate $\int_{-2}^{2}(x+3)\sqrt{4-x^2}\,\mathrm{d}x$ by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.

$\int_{-2}^{2}(x+3)\sqrt{4-x^2}\,\mathrm{d}x = 3\int_{-2}^{2}\sqrt{4-x^2}\,\mathrm{d}x + \int_{-2}^{2}x\sqrt{4-x^2}\,\mathrm{d}x$. The first integral is thrice the area of the semicircle above the x-axis $x^2+y^2=4$, which is 6π; using $u=4-x^2$ with the second interval gives $\mathrm{d}u=-2x\,\mathrm{d}x$ and $\int_{-2}^{2}x\sqrt{4-x^2}\,\mathrm{d}x = -\frac{1}{2}\int_{0}^{0}\sqrt{u}\,\mathrm{d}u = 0$. Thus, the integral is equal to 6π.

§5.5 #91 If a and b are positive numbers, show that $\int_{0}^{1}x^a(1-x)^b\,\mathrm{d}x = \int_{0}^{1}x^b(1-x)^a\,\mathrm{d}x$.

Using $y=1-x$, gives $\mathrm{d}y=-\,\mathrm{d}x$ and $\int_{0}^{1}x^a(1-x)^b\,\mathrm{d}x = -\int_{1}^{0}(1-y)^a y^b\,\mathrm{d}y = \int_{0}^{1}y^b(1-y)^a\,\mathrm{d}y = \int_{0}^{1}x^b(1-x)^a\,\mathrm{d}x$, which results from changing the dummy variable of the integral from y to x.

§7.1 #53 Use integration by parts to prove $\int \tan^n x\,\mathrm{d}x = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,\mathrm{d}x$, $(n\neq 1)$.

Noting $\int \tan x\,\mathrm{d}x = \ln|\sec x| + C$, consider for $n\geq 2$, that $\tan^n x = \tan^{n-2}x(\sec^2 x - 1)$, so $\int \tan^n x\,\mathrm{d}x = \int \tan^{n-2}x\sec^2 x\,\mathrm{d}x - \int \tan^{n-2}x\,\mathrm{d}x$. For the first integral, letting $u=\tan x$ sets $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$, such that

$$\int \tan^n x\,\mathrm{d}x = \int u^{n-2}\,\mathrm{d}u - \int \tan^{n-2}x\,\mathrm{d}x = \frac{u^{n-1}}{n-1} - \int \tan^{n-2}x\,\mathrm{d}x = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,\mathrm{d}x.$$

§7.1 #67 The Fresnel function $S(x) = \int_{0}^{x}\sin\left(\frac{1}{2}\pi t^2\right)\mathrm{d}t$ was discussed in Example 5.3.3 and is used extensively in the theory of optics. Find $\int S(x)\,\mathrm{d}x$. [Your answer will involve $S(x)$.]

Let $u=S(x)$: then $\mathrm{d}v = \mathrm{d}x$ gives $v=x$ and $\mathrm{d}u = \sin\left(\frac{1}{2}\pi x^2\right)\mathrm{d}x$, so

$$\int S(x)\,\mathrm{d}x = xS(x) - \int x\sin\left(\frac{1}{2}\pi x^2\right)\mathrm{d}x.$$

Let $w=\frac{\pi x^2}{2}$: then $\mathrm{d}w = \pi x\,\mathrm{d}x$, so

$$\int S(x)\,\mathrm{d}x = xS(x) - \frac{1}{\pi}\int \sin w\,\mathrm{d}w = xS(x) + \frac{1}{\pi}\cos w + C = xS(x) + \frac{1}{\pi}\cos\left(\frac{1}{2}\pi x^2\right) + C.$$

§7.1 #71 Suppose that $f(1)=2$, $f(4)=7$, $f'(1)=5$, $f'(4)=3$, and f'' is continuous. Find the value of $\int_{1}^{4}xf''(x)\,\mathrm{d}x$.

Letting $u=x$ sets $\mathrm{d}v = f''(x)\,\mathrm{d}x$, such that $\mathrm{d}u=\mathrm{d}x$, $v=f'(x)$ and $\int_{1}^{4}xf''(x)\,\mathrm{d}x = xf'(x)]_{x=1}^{x=4} - \int_{1}^{4}f'(x)\,\mathrm{d}x = 4f'(4)-f'(1)-f(4)+f(1) = 12-5-7+2 = 2.$

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

Name: ______________________________

This is an open-book quiz. Work quietly and individually. You may use a calculator, but not any internet-enabled devices. Make sure that there is a seat between you and your neighbors.

Evaluate the following (1 point each):

1. $\int x \sinh x^2 \, dx$

   Let $u = x^2$: then $du = 2x\, dx$ and $\int x \sinh x^2 \, dx = \frac{1}{2}\int \sinh u \, du = \frac{1}{2}\cosh u + C = \frac{1}{2}\cosh x^2 + C$.

   Check: $\frac{d}{dx}\left[\frac{1}{2}\cosh x^2 + C\right] = \frac{1}{2}(\sinh x^2)2x$.

2. $\int \cos^{-1} x \, dx$

   Let $u = \cos^{-1} x$ and $dv = \; dx$: then $du = -\frac{dx}{\sqrt{1-x^2}}$, $v = x$ and $\int \cos^{-1} x \, dx = x\cos^{-1} x + \int \frac{x\, dx}{\sqrt{1-x^2}}$. Let $w = 1 - x^2$: then $dw = -2x\, dx$ and $\int \frac{x\, dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{dw}{\sqrt{w}} = -\sqrt{w} + C$. Thus, $\int \cos^{-1} x \, dx = x\cos^{-1} x - \sqrt{1-x^2} + C$.

   Check: $\frac{d}{dx}\left[x\cos^{-1} x - \sqrt{1-x^2} + C\right] = \cos^{-1} x - \frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} = \cos^{-1} x$.

3. $\int_0^1 \left(t^3 - \frac{2}{3}t\right) e^{3t} \, dt$

   Let $u = t^3$ and $dv = e^{3t}\, dt$: then $du = 3t^2\, dt$, $v = \frac{1}{3}e^{3t}$ and $\int \left(t^3 - \frac{2}{3}t\right) e^{3t}\, dt = \frac{1}{3}t^3 e^{3t} - \int \left(t^2 + \frac{2}{3}t\right) e^{3t}\, dt$. Let $U = t^2$ and $dV = e^{3t}\, dt$: then $dU = 2t\, dt$, $V = \frac{1}{3}e^{3t}\, dt$ and $\int \left(t^2 + \frac{2}{3}t\right) e^{3t}\, dt = \frac{1}{3}t^2 e^{3t} + C$. Thus, $\int \left(t^3 - \frac{2}{3}t\right) e^{3t}\, dt = \frac{t^3 - t^2}{3}e^{3t} + C$, and $\int_0^1 \left(t^3 - \frac{2}{3}t\right) e^{3t}\, dt = 0$.

   Check: $\frac{d}{dx}\left[\frac{1}{3}(t^3 - t^2)e^{3t} + C\right] = \frac{1}{3}\left[(3t^2 - 2t)e^{3t} + (3t^3 - 3t^2)e^{3t}\right] = \frac{3t^3 - 2t}{3}e^{3t}$.

4. $\int (\ln x^3)^2 \, dx$

   Note $(\ln x^3)^2 = (3\ln x)^2 = 9(\ln x)^2$. Let $u = (\ln x)^2$ and $dv = \; dx$: then $du = \frac{2\ln x\, dx}{x}$, $v = x$ and $\int (\ln x)^2\, dx = x(\ln x)^2 - 2\int \ln x\, dx$. Let $U = \ln x$ and $dV = \; dx$: then $dU = \frac{dx}{x}$, $V = x$ and $\int \ln x\, dx = x\ln x - \int dx$. Thus, $\int (\ln x^3)^2\, dx = 9x(\ln x)^2 - 18x\ln x + 18x + C$.

   Check: $\frac{d}{dx}\left[9x(\ln x)^2 - 18x\ln x + 18x + C\right] = 9(\ln x)^2 + 18\ln x - 18\ln x - 18 + 18 = 9(\ln x)^2 = (\ln x^3)^2$.

5. $\displaystyle\int x\cosh x\,\mathrm{d}x$

Let $u = x$ and $\mathrm{d}v = \cosh x\ \mathrm{d}x$: then $\mathrm{d}u = \ \mathrm{d}x$, $v = \sinh x$ and $\displaystyle\int x\cosh x\ \mathrm{d}x = x\sinh x - \int \sinh x\ \mathrm{d}x = x\sinh x - \cosh x + C$.
Check: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[x\sinh x - \cosh x + C\right] = \sinh x + x\cosh x - \sinh x = x\cosh x$.

6. $\displaystyle\int_0^{\sin(1/2)} \sqrt{\frac{1-4(\sin^{-1}x)^2}{1-x^2}}\ \mathrm{d}x$

Let $u = \sin^{-1}x$: then $\mathrm{d}u = \dfrac{\mathrm{d}x}{\sqrt{1-x^2}}$ and $\displaystyle\int_0^{\sin(1/2)} \sqrt{\frac{1-4(\sin^{-1}x)^2}{1-x^2}}\ \mathrm{d}x = \int_0^{1/2}\sqrt{1-4u^2}\ \mathrm{d}u = 2\int_0^{1/2}\sqrt{\frac{1}{4}-x^2}\ \mathrm{d}x$. This is twice the area of the portion of the circle $x^2+y^2 = \dfrac{1}{4}$ in the first quadrant, thus $\displaystyle\int_0^{\sin(1/2)} \sqrt{\frac{1-4(\sin^{-1}x)^2}{1-x^2}}\ \mathrm{d}x = \frac{2}{4}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{8}$.

$$
\begin{aligned}
\mathit{Check}: &\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{4}\sin^{-1}(2\sin^{-1}x) + \frac{1}{2}(\sin^{-1}x)\sqrt{1-4(\sin^{-1}x)^2} + C\right] \\
=&\frac{1}{2\sqrt{[1-4(\ln x)^2](1-x^2)}} + \frac{1}{2}\sqrt{\frac{1-4(\sin^{-1}x)^2}{1-x^2}} - \frac{2(\sin^{-1}x)^2}{\sqrt{[1-4(\sin^{-1}x)^2](1-x^2)}} \\
=&\frac{1-4(\sin^{-1}x)^2}{2\sqrt{[1-4(\sin^{-1}x)^2](1-x^2)}} + \frac{1}{2}\sqrt{\frac{1-4(\sin^{-1}x)^2}{1-x^2}} = \sqrt{\frac{1-4(\sin^{-1}x)^2}{1-x^2}}
\end{aligned}
$$

$$\left[\frac{1}{4}\sin^{-1}(2\sin^{-1}x) + \frac{1}{2}(\sin^{-1}x)\sqrt{1-4(\sin^{-1}x)^2}\right]_0^{\sin(1/2)} = \frac{1}{4}\sin^{-1}(1) + \frac{1}{2}\left(\frac{1}{2}\right)\sqrt{1-4\left(\frac{1}{2}\right)^2} = \frac{\pi}{8}.$$

7. $\displaystyle\int \sin 2x\exp(\sin^2 x)\ \mathrm{d}x$

Let $u = \sin^2 x$: then $\mathrm{d}u = 2\sin x\cos x\ \mathrm{d}x = \sin 2x\ \mathrm{d}x$ and $\displaystyle\int \sin 2x\exp(\sin^2 x)\ \mathrm{d}x = \int e^u\ \mathrm{d}u = e^u + C = \exp(\sin^2 x) + C$.
Check: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\exp(\sin^2 x) + C\right] = 2\sin x\cos x\exp(\sin^2 x) = \sin 2x\exp(\sin^2 x)$.

8. $\displaystyle\int \sinh 2x\sin x\ \mathrm{d}x$

Let $u = \sinh 2x$ and $\mathrm{d}v = \sin x\ \mathrm{d}x$: then $\mathrm{d}u = 2\cosh 2x\ \mathrm{d}x$, $v = -\cos x$ and $\displaystyle\int \sinh 2x\sin x\ \mathrm{d}x = -\sinh 2x\cos x + 2\int\cosh 2x\cos x\ \mathrm{d}x$. Let $U = \cosh 2x$ and $\mathrm{d}V = \cos x\ \mathrm{d}x$: then $\mathrm{d}U = 2\sinh 2x\ \mathrm{d}x$, $V = \sin x$ and $\displaystyle\int\cosh 2x\cos x\ \mathrm{d}x = \cosh 2x\sin x - 2\int \sinh 2x\sin x\ \mathrm{d}x$. Thus,

$$
\begin{aligned}
\int \sinh 2x\sin x\ \mathrm{d}x &= -\sinh 2x\cos x + 2\cosh 2x\sin x - 4\int\sinh 2x\sin x\ \mathrm{d}x, \\
5\int \sinh 2x\sin x\ \mathrm{d}x &= 2\cosh 2x\sin x - \cos x\sinh 2x + C, \\
\int \sinh 2x\sin x\ \mathrm{d}x &= \frac{2}{5}\cosh 2x\sin x - \frac{1}{5}\cos x\sinh 2x + C.
\end{aligned}
$$

Check: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{2}{5}\cosh 2x\sin x - \dfrac{1}{5}\cos x\sinh 2x + C\right] = \dfrac{2}{5}\cosh 2x\cos x + \dfrac{4}{5}\sinh 2x\sin x + \dfrac{1}{5}\sin x\sinh 2x - \dfrac{2}{5}\cos x\cosh 2x = \sinh 2x\sin x$.

Find $\int \sin^m x \cos^n x \, \mathrm{d}x$, $m, n \geq 0$, integers.

- If m is odd
  1. Replace $\sin^{m-1} x = (1 - \cos^2 x)^{(m-1)/2}$:
  $$\int \sin^m x \cos^n x \, \mathrm{d}x = \int (1 - \cos^2 x)^{(m-1)/2} \cos^n x \sin x \, \mathrm{d}x = \int p(\cos x) \sin x \, \mathrm{d}x,$$
  where p is a polynomial.
  2. Use the substitution $u = \cos x$ and integrate: $\mathrm{d}u = -\sin x \, \mathrm{d}x$ and
  $$\int \sin^m x \cos^n x \, \mathrm{d}x = -\int p(u) \, \mathrm{d}u = -P(u) + C = -P(\cos x) + C,$$
  where P is a polynomial such that $P' = p$.
- If n is odd
  1. Replace $\cos^{n-1} x = (1 - \sin^2 x)^{(n-1)/2}$:
  $$\int \sin^m x \cos^n x \, \mathrm{d}x = \int \sin^m x (1 - \sin^2 x)^{(n-1)/2} \cos x \, \mathrm{d}x = \int p(\sin x) \cos x \, \mathrm{d}x,$$
  where p is a polynomial.
  2. Use the substitution $u = \sin x$ and integrate: $\mathrm{d}u = \cos x \, \mathrm{d}x$ and
  $$\int \sin^m x \cos^n x \, \mathrm{d}x = \int p(u) \, \mathrm{d}u = P(u) + C = P(\sin x) + C,$$
  where P is a polynomial such that $P' = p$.
- Otherwise (m and n are both even)

  1a. If $m \geq n$: replace $\sin^n x \cos^n x = \left(\frac{\sin 2x}{2}\right)^n$ and $\sin^{m-n} x = \left(\frac{1 - \cos 2x}{2}\right)^{(m-n)/2}$
  $$\int \sin^m x \cos^n x \, \mathrm{d}x = \frac{1}{2^{(m+n)/2}} \int \sin^n 2x (1 - \cos 2x)^{(m-n)/2} \, \mathrm{d}x.$$
  1b. If $m < n$: replace $\sin^m x \cos^m x = \left(\frac{\sin 2x}{2}\right)^m$ and $\cos^{n-m} x = \left(\frac{1 + \cos 2x}{2}\right)^{(n-m)/2}$
  $$\int \sin^m x \cos^n x \, \mathrm{d}x = \frac{1}{2^{(m+n)/2}} \int \sin^m 2x (1 + \cos 2x)^{(n-m)/2} \, \mathrm{d}x.$$
  2. Use substitution $u = 2x$ and perform termwise integration (`return to top`): $\mathrm{d}u = 2 \, \mathrm{d}x$,
  $$\int \sin^m x \cos^n x \, \mathrm{d}x = \frac{1}{2^{(m+n+2)/2}} \sum_{k=0}^{|n-m|/2} \operatorname{sign}(n-m)^k \binom{|n-m|/2}{k} \int \sin^{\min(n,m)} u \cos^k u \, \mathrm{d}u.$$
  Since the power of $\sin x$ will still be even, terms where k is odd can be solved by the second case above, but this case will be revisited when k is even.

Find $\int \frac{\sin^m x}{\cos^n x} \, \mathrm{d}x$, $m, n \geq 0$, integers. (Analogous process for $\int \frac{\cos^m x}{\sin^n x} \, \mathrm{d}x$, using cofunctions.)

- If $m > n$

1. Replace $\sin^m x = (1-\cos^2 x)^{\lfloor m/2 \rfloor} \sin^{m \mod 2} x$
$$\int \frac{\sin^m x}{\cos^n x}\,\mathrm{d}x = \int \frac{(1-\cos^2 x)^{\lfloor m/2 \rfloor} \sin^{m \mod 2} x}{\cos^n x}\,\mathrm{d}x.$$
2. Perform termwise integration (`go to next cases` or `use previous algorithm`)
$$\int \frac{\sin^m x}{\cos^n x}\,\mathrm{d}x = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \binom{\lfloor m/2 \rfloor}{k} \int \sin^{m \mod 2} x \cos^{2k-n} x\,\mathrm{d}x.$$
If $2k-n \geq 0$, the previous algorithm is used, otherwise the next cases are used.

- If $m=n$, $\dfrac{\sin^n x}{\cos^n x} = \tan^n x$: Use $\displaystyle\int \tan x\,\mathrm{d}x = \int \frac{\sin x\,\mathrm{d}x}{\cos x} = -\int \frac{\mathrm{d}(\cos x)}{\cos x} = \ln|\sec x| + C$, and,
$$\int \tan^n x\,\mathrm{d}x = \int \tan^{n-2} x \sec^2 x\,\mathrm{d}x - \int \tan^{n-2} x\,\mathrm{d}x = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,\mathrm{d}x, \text{ for } n>1.$$
- Otherwise $\left(n > m, \dfrac{\sin^m x}{\cos^n x} = \tan^m x \sec^{n-m} x\right)$
  - If $n-m$ is even
    1. Replace $\sec^{n-m-2} x = (1+\tan^2 x)^{(n-m-2)/2}$:
    $$\int \frac{\sin^m x}{\cos^n x}\,\mathrm{d}x = \int \tan^m x(1+\tan^2 x)^{(n-m-2)/2} \sec^2 x\,\mathrm{d}x = \int p(\tan x)\sec^2 x\,\mathrm{d}x,$$
    where p is a polynomial.
    2. Use the substitution $u = \tan x$ and integrate: $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$ and
    $$\int \frac{\sin^m x}{\cos^n x}\,\mathrm{d}x = \int p(u)\,\mathrm{d}u = P(u) + C = P(\tan x) + C,$$
    where P is a polynomial such that $P' = p$.
  - If m is odd
    1. Replace $\tan^{m-1} x = (\sec^2 x - 1)^{(m-1)/2}$:
    $$\int \frac{\sin^m x}{\cos^n x}\,\mathrm{d}x = \int (\sec^2 x - 1)^{(m-1)/2} \sec^{n-m} x \tan x\,\mathrm{d}x = \int p(\sec x)\sec x \tan x\,\mathrm{d}x,$$
    where p is a polynomial.
    2. Use the substitution $u = \sec x$ and integrate: $\mathrm{d}u = \sec x \tan x\,\mathrm{d}x$ and
    $$\int \frac{\sin^m x}{\cos^n x}\,\mathrm{d}x = \int p(u)\,\mathrm{d}u = P(u) + C = P(\sec x) + C,$$
    where P is a polynomial such that $P' = p$.
  - Otherwise (m is even and n is odd)
    1. Replace $\tan^m x = (\sec^2 x - 1)^{m/2}$:
    $$\int \frac{\sin^m x}{\cos^n x}\,\mathrm{d}x = \int (\sec^2 x - 1)^{m/2} \sec^{n-m} x\,\mathrm{d}x = \sum_{k=0}^{m/2} (-1)^k \binom{m/2}{k} \int \sec^{n-2k} x\,\mathrm{d}x.$$
    2. Use $\displaystyle\int \sec x\,\mathrm{d}x = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,\mathrm{d}x = \int \frac{\mathrm{d}(\sec x + \tan x)}{\sec x + \tan x} = \ln|\sec x + \tan x| + C$,
    and, for $n > 0$, by integration by parts ($u = \sec^{2n-1} x$ and $\mathrm{d}v = \sec^2 x\,\mathrm{d}x$):
    $$\begin{aligned}\int \sec^{2n+1} x\,\mathrm{d}x &= \sec^{2n-1} x \tan x - (2n-1)\int \sec^{2n-1} x \tan^2 x\,\mathrm{d}x \\ &= \sec^{2n-1} x \tan x - (2n-1)\int \sec^{2n+1} x\,\mathrm{d}x + (2n-1)\int \sec^{2n-1} x\,\mathrm{d}x \\ &= \frac{1}{2n}\sec^{2n-1} x \tan x + \frac{2n-1}{2n}\int \sec^{2n-1} x\,\mathrm{d}x.\end{aligned}$$

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§7.2 #56 Evaluate $\int \sin x \cos x \, \mathrm{d}x$ by four methods:

(a) the substitution $u = \cos x$

This gives $\mathrm{d}u = -\sin x \, \mathrm{d}x$, thus $\int \sin x \cos x \, \mathrm{d}x = -\int u \, \mathrm{d}u = -\frac{u^2}{2} + C = -\frac{\cos^2 x}{2} + C$.

(b) the substitution $u = \sin x$

This gives $\mathrm{d}u = \cos x \, \mathrm{d}x$, thus $\int \sin x \cos x \, \mathrm{d}x = \int u \, \mathrm{d}u = \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C$.

(c) the identity $\sin 2x = 2 \sin x \cos x$

Letting $u = 2x$ gives $\mathrm{d}u = 2 \, \mathrm{d}x$, thus $\int \sin x \cos x \, \mathrm{d}x = \frac{1}{4} \int \sin u \, \mathrm{d}u = -\frac{\cos u}{4} + C = -\frac{\cos 2x}{4} + C$.

(d) integration by parts

Letting $u = \cos x$ sets $\mathrm{d}v = \sin x \, \mathrm{d}x$, thus $\mathrm{d}u = -\sin x \, \mathrm{d}x$, $v = -\cos x$ and $\int \sin x \cos x \, \mathrm{d}x = -\cos^2 x - \int \sin x \cos x \, \mathrm{d}x$, so $\int \sin x \cos x \, \mathrm{d}x = -\frac{\cos^2 x}{2} + C$.

Letting $u = \sin x$ sets $\mathrm{d}v = \cos x \, \mathrm{d}x$, thus $\mathrm{d}u = \cos x \, \mathrm{d}x$, $v = \sin x$ and $\int \sin x \cos x \, \mathrm{d}x = \sin^2 x - \int \sin x \cos x \, \mathrm{d}x$, so $\int \sin x \cos x \, \mathrm{d}x = \frac{\sin^2 x}{2} + C$.

Explain the different appearances of the answers.

From the identities $\cos 2x = \cos^2 x - \sin^2 x$ and $\sin^2 x + \cos^2 x = 1$, $-\cos 2x = 1 - 2\cos^2 x = 2\sin^2 x - 1$, so the arbitrary constant C absorbs the constant $\pm\frac{1}{4}$ in the equivalent values for $-\frac{\cos 2x}{4}$, and the expressions are equivalent.

§7.2 #70 A *finite Fourier sine series* is given by the sum $f(x) = \sum_{n=1}^{N} a_n \sin nx = a_1 \sin x + a_2 \sin 2x + \cdots + a_N \sin Nx$. Show that the mth coefficient a_m is given by the formula $a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, \mathrm{d}x$.

First, it must be shown that $\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n, \end{cases}$ where m and n are positive integers: $2 \sin mx \sin nx = \cos[(m-n)x] - \cos[(m+n)x]$, so, if $m \neq n$,

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x = \frac{1}{2} \int_{-\pi}^{\pi} (\cos[(m-n)x] - \cos[(m+n)x]) \, \mathrm{d}x = \left[\frac{\sin[(m-n)x]}{2(m-n)} - \frac{\sin[(m+n)x]}{2(m+n)} \right]_{-\pi}^{\pi},$$

all of whose terms are zero. But, if $m = n$,

$$\int_{-\pi}^{\pi} \sin^2 nx \, \mathrm{d}x = \frac{1}{2} \int_{-\pi}^{\pi} \mathrm{d}x - \frac{1}{2} \int_{-\pi}^{\pi} \cos 2nx \, \mathrm{d}x = \left[\frac{x}{2} - \frac{\sin 2nx}{4n} \right]_{-\pi}^{\pi} = \pi.$$

Then,

$$\int_{-\pi}^{\pi} f(x) \sin mx \, \mathrm{d}x = \int_{-\pi}^{\pi} \left(\sum_{n=1}^{N} a_n \sin nx \sin mx \right) \mathrm{d}x = \sum_{n=1}^{N} \left[a_n \int_{-\pi}^{\pi} \sin nx \sin mx \, \mathrm{d}x \right]$$
$$= 0 + \cdots + 0 + a_m \int_{-\pi}^{\pi} \sin mx \sin mx \, \mathrm{d}x + 0 + \cdots = a_m \pi,$$

and the conclusion follows from dividing both sides by π.

§7.3 #32 Evaluate $\displaystyle\int \frac{x^2}{(x^2+a^2)^{3/2}}\,\mathrm{d}x$

(a) by trigonometric substitution
Using $x = a\tan\theta$ gives $\mathrm{d}x = a\sec^2\theta\,\mathrm{d}\theta$, $\sqrt{x^2+a^2} = a\sec\theta$ and

$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\,\mathrm{d}x = \int \frac{a^2\tan^2\theta(a\sec^2\theta\,\mathrm{d}\theta)}{a^3\sec^3\theta} = \int \frac{\sin^2\theta\,\mathrm{d}\theta}{\cos\theta} = \int \frac{(1-\cos^2\theta)\,\mathrm{d}\theta}{\cos\theta}$$
$$= \int \sec\theta\,\mathrm{d}\theta - \int \cos\theta\,\mathrm{d}\theta = \ln|\sec\theta+\tan\theta| - \sin\theta + C$$
$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| - \frac{x/a}{\sqrt{x^2+a^2}/a} + C = \ln\left|x+\sqrt{x^2+a^2}\right| - \frac{x}{\sqrt{x^2+a^2}} + C.$$

(b) by the hyperbolic substitution $x = a\sinh t$
Using $x = a\sinh t$ gives $\mathrm{d}x = a\cosh t\,\mathrm{d}t$, $\sqrt{x^2+a^2} = a\cosh t$ and

$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\,\mathrm{d}x = \int \frac{a^2\sinh^2 t(a\cosh t\,\mathrm{d}t)}{a^3\cosh^3 t} = \int \tanh^2 t\,\mathrm{d}t = \int (1-\operatorname{sech}^2 t)\,\mathrm{d}t,$$

since dividing both sides of the identity $\cosh^2 t - \sinh^t = 1$ by $\cosh^2 t$ gives $1-\tanh^2 t = \operatorname{sech}^2 t$,

$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\,\mathrm{d}x = \int \mathrm{d}t - \int \operatorname{sech}^2 t\,\mathrm{d}t = t - \tanh t + C = \sinh^{-1}\left(\frac{x}{a}\right) - \frac{x/a}{\sqrt{x^2+a^2}/a} + C,$$

since $\dfrac{\mathrm{d}}{\mathrm{d}t}\tanh t = \dfrac{\cosh^2 t - \sinh^2 t}{\cosh^2 t} = \operatorname{sech}^2 t$, and

$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\,\mathrm{d}x == \ln\left|\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2+1}\right| - \frac{x}{\sqrt{x^2+a^2}} + C = \ln\left|x+\sqrt{x^2+a^2}\right| - \frac{x}{\sqrt{x^2+a^2}} + C,$$

since $y = \sinh^{-1}x$ means $x = \sinh y = \dfrac{e^y - e^{-y}}{2}$ and $e^{2y} - 2xe^y - 1 = 0$: by the quadratic formula, $e^y = \dfrac{2x}{2} \pm \dfrac{\sqrt{(-2x)^2+4}}{4} = x + \sqrt{x^2+1} \geq 0$, thus $y = \ln|x+\sqrt{x^2+1}| = \sinh^{-1}x$.

§7.3 #44 A water storage tank has the shape of a cylinder with diameter 10 m. It is mounted so that the circular cross-sections are vertical. If the depth of the water is 7 m, what percentage of the total capacity is being used?

The percentage of the total capacity occupied by the water in the tank is the same as the area of any vertical circular cross-section that is in water: if such a cross-section has a cartesian grid with units of length one meter, the origin in the center of the circle, and whose positive x-axis is pointing downward, the percentage to be determined is the area of the portion of the circle $x^2+y^2=25$ to the right of the line $x=-2$. By symmetry with respect to the x-axis, the percentage is the area under the curve $y=\sqrt{25-x^2}$ from $x=-2$ to $x=5$, divided by the area of the semicircle, which is $\dfrac{25\pi}{2}$. Thus, using $x = 5\sin\theta$, giving $\mathrm{d}x = 5\cos\theta\,\mathrm{d}\theta$ and $\sqrt{25-x^2} = 5\cos\theta$, the percentage is

$$\frac{2}{25\pi}\int_{-2}^{5}\sqrt{25-x^2}\,\mathrm{d}x = \frac{2}{\pi}\int_{\sin^{-1}(-2/5)}^{\pi/2}\cos^2\theta\,\mathrm{d}\theta = \frac{1}{\pi}\int_{\sin^{-1}(-2/5)}^{\pi/2}\mathrm{d}\theta + \frac{1}{\pi}\int_{\sin^{-1}(-2/5)}^{\pi/2}\cos 2\theta\,\mathrm{d}\theta$$
$$= \frac{1}{2\pi}\left[2\theta+\sin 2\theta\right]_{\sin^{-1}(-2/5)}^{\pi/2} = \frac{1}{2} - \frac{1}{\pi}\sin^{-1}\left(-\frac{2}{5}\right) + 0 - \frac{1}{2\pi}\sin\left[2\sin^{-1}\left(-\frac{2}{5}\right)\right],$$

where $\cos\left[\sin\left(-\dfrac{2}{5}\right)\right] = \dfrac{\sqrt{21}}{5}$ gives the last term as $2\left(-\dfrac{2}{5}\right)\dfrac{\sqrt{21}}{5} = \dfrac{-4\sqrt{21}}{25}$; thus the percentage is $\dfrac{25\pi+4\sqrt{21}}{50\pi} - \dfrac{1}{\pi}\sin^{-1}\left(-\dfrac{2}{5}\right) \approx 0.7476842$ or about 74.8%.

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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§7.2 #65 A particle moves on a straight line with velocity function $v(t) = \sin\omega t\cos^2\omega t$. Find its position function $s = f(t)$ if $f(0) = 0$.

Since $s(t) = \int_0^t v(x)\,\mathrm{d}x$, as $s(0) = 0$, using $u = \cos\omega x$ gives $\mathrm{d}u = -\omega\sin\omega x\,\mathrm{d}x$ and

$$s(t) = \int_0^t \sin\omega x\cos^2\omega x\,\mathrm{d}x = -\frac{1}{\omega}\int_1^{\cos\omega t} u^2\,\mathrm{d}u = \left[\frac{u^3}{3\omega}\right]_{\cos\omega t}^1 = \frac{1-\cos^3\omega t}{3\omega},$$

assuming $\omega \neq 0$, otherwise $v(t) = s(t) \equiv 0$.

§7.3 #13 Evaluate $\int \frac{\sqrt{x^2-9}}{x^3}\,\mathrm{d}x$.

Using $x = 3\sec\theta$ gives $\mathrm{d}x = 3\sec\theta\tan\theta\,\mathrm{d}\theta$ and $\sqrt{x^2-9} = 3\tan\theta$ and, since $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$,

$$\int \frac{\sqrt{x^2-9}}{x^3}\,\mathrm{d}x = \int \frac{3\tan\theta(3\sec\theta\tan\theta\,\mathrm{d}\theta)}{(3\sec\theta)^3}\,\mathrm{d}x = \frac{1}{3}\int\sin^2\theta\,\mathrm{d}\theta = \frac{1}{6}\int\mathrm{d}\theta - \frac{1}{6}\int\cos 2\theta\,\mathrm{d}\theta$$
$$= \frac{\theta}{6} - \frac{\sin 2\theta}{12} + C = \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C.$$

Check: $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C\right] = \frac{1}{6x\sqrt{(x/3)^2-1}} - \frac{2x^2(x/\sqrt{x^2-9}) - 4x\sqrt{x^2-9}}{4x^4} =$ $\frac{1}{2x\sqrt{x^2-9}} - \frac{x^3-2x(x^2-9)}{2x^4\sqrt{x^2-9}} = \frac{x^3-x^3+2x^3-18x}{2x^4\sqrt{x^2-9}} = \frac{x^2-9}{x^3\sqrt{x^2-9}} = \frac{\sqrt{x^2-9}}{x^3}.$

§7.3 #29 Evaluate $\int x\sqrt{1-x^4}\,\mathrm{d}x$.

Letting $u = x^2 = \sin\theta$ gives $\mathrm{d}u = 2x\,\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$, $\sqrt{1-u^2} = \sqrt{1-x^4} = \cos\theta$ and $\int x\sqrt{1-x^4}\,\mathrm{d}x =$ $\frac{1}{2}\int\cos^2\theta\,\mathrm{d}\theta = \frac{1}{4}\int\mathrm{d}\theta + \frac{1}{4}\int\cos 2\theta\,\mathrm{d}\theta = \frac{\theta}{4} + \frac{\sin 2\theta}{8} + C = \frac{1}{4}\sin^{-1}(x^2) + \frac{x^2\sqrt{1-x^4}}{4} + C.$

Check: $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{4}\sin^{-1}(x^2) + \frac{x^2\sqrt{1-x^4}}{4} + C\right] = \frac{x}{2\sqrt{1-x^4}} + \frac{x\sqrt{1-x^4}}{2} - \frac{x^5}{2\sqrt{1-x^4}} = \frac{x+x(1-x^4)-x^5}{2\sqrt{1-x^4}} =$ $\frac{x(1-x^4)}{\sqrt{1-x^4}} = x\sqrt{1-x^4}.$

§7.3 #31 (a) Use trigonometric substitution to show that $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) + C$.

Using $x = a\tan\theta$ gives $\mathrm{d}x = a\sec^2\theta\,\mathrm{d}\theta$, $\sqrt{x^2+a^2} = a\sec\theta$ and $\int\frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \int\frac{a\sec^2\theta\,\mathrm{d}\theta}{a\sec\theta} =$ $\int\sec\theta\,\mathrm{d}\theta = \ln|\sec\theta+\tan\theta| + C = \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C = \ln(x+\sqrt{x^2+a^2}) + C.$

(b) Use the hyperbolic substitution $x = a\sinh t$ to show that $\int\frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$.

Using $x = a\sinh t$ gives $\mathrm{d}x = a\cosh t\,\mathrm{d}t$, $\sqrt{x^2+a^2} = a\cosh t$ and $\int\frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \int\frac{a\cosh t\,\mathrm{d}t}{a\cosh t} =$ $\int\mathrm{d}t = t + C = \sinh^{-1}\left(\frac{x}{a}\right) + C.$

These formulas are connected by $\sinh^{-1}x = \ln(x+\sqrt{x^2+1})$, $x\in\mathbb{R}$.

Note $\sinh^{-1}\left(\frac{x}{a}\right) = \ln\left(\frac{x}{a} + \sqrt{\left[\frac{x}{a}\right]^2+1}\right) = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right|.$

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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Evaluate the integral.

§7.5 #26 $\displaystyle\int_0^1 \frac{(3x^2+1)\,\mathrm{d}x}{x^3+x^2+x+1}$

Since the denominator can be factored into $(x^2+1)(x+1)$, the integrand can be split into

$$\int_0^1 \frac{3x^2+1}{x^3+x^2+x+1}\,\mathrm{d}x = \int_0^1 \frac{A\,\mathrm{d}x}{x+1} + \int_0^1 \frac{Bx+C}{x^2+1}\,\mathrm{d}x = \left[A\ln|x+1| + \frac{B}{2}\ln(x^2+1) + C\tan^{-1}x\right]_0^1$$
$$= \frac{2A+B}{B}\ln 2 + \frac{C\pi}{4},\ \text{where } 3x^2+1 = A(x^2+1)+(Bx+C)(x+1).$$

The unknowns A, B and C are defined by a system: $A+B=3$, $B+C=0$ and $A+C=1$, so $A=2$, $B=1$ and $C=-1$, so $\displaystyle\int_0^1 \frac{(3x^2+1)\,\mathrm{d}x}{x^3+x^2+x+1} = \frac{5}{2}\ln 2 - \frac{\pi}{4}$.

Check: $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left((x+1)^2\sqrt{x^2+1}\right) - \tan^{-1}x\right] = \frac{2(x+1)\sqrt{x^2+1} + \dfrac{x(x+1)^2}{\sqrt{x^2+1}}}{(x+1)^2\sqrt{x^2+1}} - \frac{1}{x^2+1}$$
$$= \frac{2(x^2+1)+x(x+1)-(x+1)}{(x+1)(x^2+1)} = \frac{2x^2+2+x^2+x-x-1}{(x+1)(x^2+1)} = \frac{3x^2+1}{x^3+x^2+x+1}$$

§7.5 #52 $\displaystyle\int \frac{\mathrm{d}x}{x(x^4+1)}$

Since the denominator can be factored into $x(x^2+\sqrt{2}+1)(x^2-\sqrt{2}+1)$, the integrand can be split into

$$\int \frac{\mathrm{d}x}{x(x^4+1)}\,\mathrm{d}x = \int \frac{A\,\mathrm{d}x}{x} + \int \frac{Bx+C}{[x+(1/\sqrt{2})]^2+(1/2)}\,\mathrm{d}x + \int \frac{Dx+E}{[x-(1/\sqrt{2})]^2+(1/2)}\,\mathrm{d}x$$
$$= A\ln|x| + \int \frac{Bu_+ - (B/\sqrt{2}) + C}{u_+^2+(1/\sqrt{2})^2}\,\mathrm{d}x + \int \frac{Du_- + (D/\sqrt{2}) + E}{u_-^2+(1/\sqrt{2})^2}\,\mathrm{d}x$$
$$= A\ln|x| + \frac{B}{2}\ln(x^2+\sqrt{2}x+1) + (\sqrt{2}C-B)\tan^{-1}(\sqrt{2}x+1) + \frac{D}{2}\ln(x^2-\sqrt{2}x+1)$$
$$+ (\sqrt{2}E+D)\tan^{-1}(\sqrt{2}x-1),\ \text{where } u_\pm = x \pm \frac{1}{\sqrt{2}},\ \text{and the unknowns satisfy}$$
$$1 = A(x^4+1) + (Bx+C)x(x^2-\sqrt{2}x+1) + (Dx+E)x(x^2+\sqrt{2}x+1).$$

The unknowns A, B, C, D and E are defined by a system: $A+B+D=0$, $-\sqrt{2}B+C+\sqrt{2}D+E=0$, $B-\sqrt{2}C+D+\sqrt{2}E=0$, $C+E=0$ and $A=1$, so $B=-\dfrac{1}{2}$, $C=-\dfrac{1}{2\sqrt{2}}$, $D=-\dfrac{1}{2}$ and $E=\dfrac{1}{2\sqrt{2}}$, so $\displaystyle\int \frac{\mathrm{d}x}{x(x^4+1)} = \ln|x| - \frac{1}{4}\ln(x^4+1) + C$.

Check: $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left|\frac{x}{\sqrt[4]{x^4+1}}\right|\right] = \frac{\sqrt[4]{x^4+1}}{x}\,\frac{\sqrt[4]{x^4+1} - \dfrac{x^4}{\sqrt[4]{(x^4+1)^3}}}{\sqrt{x^4+1}} = \frac{x^4+1-x^4}{x(x^4+1)} = \frac{1}{x(x^4+1)}$$

§7.5 #56 $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x}+x\sqrt{x}}$

Using $u^2=x$ gives $2u\,\mathrm{d}u = \mathrm{d}x$ and $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x}+x\sqrt{x}} = \int \frac{2u\,\mathrm{d}u}{u+u^3} = 2\tan^{-1}u + 1 = 2\tan^{-1}\sqrt{x} + C$.

Check: $$\frac{\mathrm{d}}{\mathrm{d}x}\left[2\tan^{-1}\sqrt{x}\right] = \frac{2}{x+1}\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}+x\sqrt{x}}$$

§7.5 #72 $\displaystyle\int \frac{\ln(x+1)\,\mathrm{d}x}{x^2}$

Using $u = \ln(x+1)$ leaves $\mathrm{d}v = \dfrac{\mathrm{d}x}{x^2}$, thus $\mathrm{d}u = \dfrac{\mathrm{d}x}{x+1}$ and $v = -\dfrac{1}{x}$, and

$$\int \frac{\ln(x+1)\,\mathrm{d}x}{x^2} = -\frac{\ln(x+1)}{x} + \int \frac{\mathrm{d}x}{x(x+1)} = -\frac{\ln(x+1)}{x} + \int \frac{\mathrm{d}x}{x} - \int \frac{\mathrm{d}x}{x+1} = \ln\left|\frac{x}{x+1}\right| - \frac{\ln(x+1)}{x} + C.$$

Check: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left|\dfrac{x}{x+1}\right| - \dfrac{\ln(x+1)}{x}\right] = \dfrac{x+1}{x}\dfrac{1}{(x+1)^2} - \dfrac{\dfrac{x}{x+1} - \ln(x+1)}{x^2} = \dfrac{1}{x(x+1)} - \dfrac{1}{x(x+1)} + \dfrac{\ln(x+1)}{x^2}$

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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This is an open-book quiz. Work quietly and individually. You may use a calculator, but not any internet-enabled devices. Make sure that there is a seat between you and your neighbors.

Evaluate the following (2 points each):

1. $\displaystyle\int \frac{1+\sqrt{(x-1)^3}}{x^2(x-1)}\,\mathrm{d}x$

   $\displaystyle\int \frac{1+\sqrt{(x-1)^3}}{x^2(x-1)}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{x^2(x-1)} + \int \frac{\sqrt{x-1}}{x^2}\,\mathrm{d}x$. For the first integral:

$$\int \frac{\mathrm{d}x}{x^2(x-1)} = A\int \frac{\mathrm{d}x}{x} + B\int \frac{\mathrm{d}x}{x^2} + C\int \frac{\mathrm{d}x}{x-1} = A\ln|x| - \frac{B}{x} + C\ln|x-1| + D,$$

   where $1 = Ax(x-1) + B(x-1) + Cx^2 = (A+C)x^2 + (B-A)x - B$, so $B = -1 = A$, $C = 1$. For the second integral: using $u = \sqrt{x-1}$ gives $\mathrm{d}u = \dfrac{\mathrm{d}x}{2\sqrt{x-1}}$, $x = u^2+1$ and

$$\int \frac{\sqrt{x-1}}{x^2}\,\mathrm{d}x = 2\int \frac{(\sqrt{x-1})^2}{x^2}\frac{\mathrm{d}x}{2\sqrt{x-1}} = 2\int \frac{u^2}{(u^2+1)^2}\,\mathrm{d}u = 2\int \frac{Au+B}{u^2+1}\,\mathrm{d}u + 2\int \frac{Cu+D}{(u^2+1)^2}\,\mathrm{d}u,$$

   where $u^2 = (Au+B)(u^2+1) + Cu + D = Au^3 + Bu^2 + (A+C)u + (B+D)$, so $A = C = 0$, $B = 1$ and $D = -1$. Using $u = \tan\theta = \sqrt{x-1}$ gives $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$, $u^2+1 = \sec^2\theta = x$ and

$$\int \frac{\sqrt{x-1}}{x^2}\,\mathrm{d}x = 2\int \frac{\mathrm{d}u}{u^2+1} - 2\int \frac{\mathrm{d}u}{(u^2+1)^2} = 2\int \mathrm{d}\theta - 2\int \cos^2\theta\,\mathrm{d}\theta = \int \mathrm{d}\theta - \int \cos 2\theta\,\mathrm{d}\theta$$
$$= \theta - \sin\theta\cos\theta + D = \tan^{-1}\sqrt{x-1} - \frac{\sqrt{x-1}}{x} + D,$$

   since $\sin\theta\cos\theta = \dfrac{\tan\theta}{\sec^2\theta}$. Thus,

$$\int \frac{1+\sqrt{(x-1)^3}}{x^2(x-1)}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{x^2(x-1)} + \int \frac{\sqrt{x-1}}{x^2}\,\mathrm{d}x = \ln\left|\frac{x-1}{x}\right| + \tan^{-1}\sqrt{x-1} + \frac{1-\sqrt{x-1}}{x} + C.$$

2. $\displaystyle\int \csc x\cot^2 x\,\mathrm{d}x$

   Since $\cot^2 x + 1 = \csc^2 x$, $\displaystyle\int \csc x\cot^2 x\,\mathrm{d}x = \int \csc^3 x\,\mathrm{d}x - \int \csc x\,\mathrm{d}x$. For the first integral, using $u = \csc x$ gives $\mathrm{d}v = \csc^2 x\,\mathrm{d}x$, $\mathrm{d}u = -\csc x\cot x\,\mathrm{d}x$, $v = -\cot x$ and

$$\int \csc^3 x\,\mathrm{d}x = -\cot x\csc x - \int \csc x\cot^2 x\,\mathrm{d}x = \int \csc x\,\mathrm{d}x - \int \csc^3 x\,\mathrm{d}x - \cot x\csc x,$$

   thus $\displaystyle\int \csc^3 x\,\mathrm{d}x = \frac{1}{2}\int \csc x\,\mathrm{d}x - \frac{1}{2}\cot x\csc x$. For the second integral,

$$\int \csc x\,\mathrm{d}x = \int \csc x\frac{\csc x+\cot x}{\csc x+\cot x}\,\mathrm{d}x = -\int \frac{\mathrm{d}(\csc x+\cot x)}{\csc x+\cot x} = -\ln|\csc x+\cot x| + C.$$

   Thus, $\displaystyle\int \csc x\cot^2 x\,\mathrm{d}x = -\frac{1}{2}\left[\cot x\csc x + \ln|\csc x+\cot x|\right] + C.$

3. $\displaystyle\int_0^1 x^2\sqrt{x^2+1}\,\mathrm{d}x$

Using $x = \tan\theta$ gives $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$, $\sqrt{x^2+1} = \sec\theta$ and, since $\sec^2 x = 1+\tan^2 x$,

$$\int_0^1 x^2\sqrt{x^2+1}\,\mathrm{d}x = \int_0^{\pi/4} \tan^2\theta\sec^3\theta\,\mathrm{d}\theta = \int_0^{\pi/4} \sec^5\theta\,\mathrm{d}\theta - \int_0^{\pi/4}\sec^3\theta\,\mathrm{d}\theta.$$

For the first integral, the reduction formula gives $\displaystyle\int \sec^5\theta\,\mathrm{d}\theta = \frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{4}\int \sec^3\theta\,\mathrm{d}\theta$. For the second integral, the reduction formula gives $\displaystyle\int\sec^3\theta\,\mathrm{d}\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\,\mathrm{d}\theta$, where $\displaystyle\int\sec\theta\,\mathrm{d}\theta = \ln|\sec\theta+\tan\theta| + C$. Thus,

$$\begin{aligned}\int_0^1 x^2\sqrt{x^2+1}\,\mathrm{d}x &= \left[\frac{1}{4}\sec^3\theta\tan\theta - \frac{1}{8}\sec\theta\tan\theta - \frac{1}{8}\ln|\sec\theta+\tan\theta|\right]_0^{\pi/4} \\ &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8} - \frac{1}{8}\ln|1+\sqrt{2}| = \frac{3\sqrt{8}+\ln|\sqrt{2}-1|}{8}.\end{aligned}$$

4. $\displaystyle\int\frac{\mathrm{d}x}{\sin x(1-\cos x)}$

Using $t = \tan\left(\dfrac{x}{2}\right)$ gives $\cos x = \dfrac{1-t^2}{1+t^2}$, $\sin x = \dfrac{2t}{1+t^2}$, $\mathrm{d}x = \dfrac{2\,\mathrm{d}t}{1+t^2}$ and

$$\begin{aligned}\int\frac{\mathrm{d}x}{\sin x(1-\cos x)} &= \int\frac{2\,\mathrm{d}t/(1+t^2)}{2t(1-[(1-t^2)/(1+t^2)])/(1+t^2)} = \int\frac{\mathrm{d}t}{t(2t^2)/(1-t^2)} = \frac{1}{2}\int\frac{1+t^2}{t^3}\,\mathrm{d}t \\ &= \frac{1}{2}\int\frac{\mathrm{d}t}{t^3} + \frac{1}{2}\int\frac{\mathrm{d}t}{t} = \frac{1}{2}\ln t - \frac{1}{4t^2} + C = \frac{1}{2}\ln\left[\tan\left(\frac{x}{2}\right)\right] - \frac{1}{4}\cot\left(\frac{x}{2}\right) + C.\end{aligned}$$