MAT 200–Calculus and Analytic Geometry II

Fall 2018

Prerequisites: MAT 150 or MAT 180 or SEM 1101

General Information

Class Schedule: Tuesdays and Thursdays 9:00–10:50am
Classroom: Plato (SR 3D)
Instructor: Michael Daniel Samson
Contact: `mat200@mdvsamson.work`, +65 6577 1944
Class Webpage: Moodle, `mdvsamson.work/mat200`
Office Hours: Tuesdays and Thursdays 11:00–12:00pm, Mondays 2:00–7:00pm or by appointment (through email)

Description

This course builds on the introduction to calculus in MAT 150. Topics in integration include applications of the integral in physics and geometry, and techniques of integration. The course also covers sequences and series of real numbers, power series and Taylor series, and calculus of transcendental functions. Further topics may include a basic introduction to concepts in multivariable and vector calculus.

Course Objectives and Learning Outcomes

Upon completing this course students should be able to:

- Understand the concept of definite integral as a limit of Riemann sums.
- Find indefinite and improper integrals using different integration techniques.
- Perform standard operations with convergent power series, find Taylor and Maclaurin representations.
- Use integrals to solve applied problems and analyze graphs of curves.

Textbooks

CALCULUS Early Transcendentals 8e, International Metric Version, James Stewart, Cengage Learning, ISBN-10 1-305-27237-4, ISBN-13 978-1-305-27237-8

Optional Textbooks

CALCULUS Early Vectors, James Stewart, Brooks/Cole Cengage Learning, ISBN-10 0-534-34941-2, ISBN-13 978-0-534-49348-6

Outline and Tentative Dates

The following schedule is subject to change.

Integration and Some Techniques
September 4: Summations
September 6: Fundamental Theorem of Calculus
September 11: Substitution Rule, **quiz**
September 13: Integration by Parts
September 18: Trigonometric Integrals
September 20: Trigonometric Substitution
September 25: Partial Fractions, **quiz**
September 27: ⋆ Improper Integrals
October 2: **Examination** (discussion on October 4)
Approximation of Definite Integrals as Infinite Sums
October 9: Approximation of Integrals
October 11: Sequences
October 16: Series
October 18: Tests of Series Convergence
October 23: Truncation Error, **quiz**
October 25: Power Series
October 30: Taylor and Maclaurin Series, **quiz**
November 1: Applications of Taylor Series
November 6: *Deepavali*
November 8: **Examination** (discussion on November 13)
Geometric and Physical Applications of Integration
November 15: Areas Between Curves,Volumes
November 20: Cylindrical Shells, **quiz**
November 24: Work, ⋆ Average Value of a Function
November 27: Arc Length
November 29: Areas of Surfaces of Revolution
December 4: Applications in Other Fields, **quiz**
December 6: ⋆ Further Applications of Integration
December 10–14: **Examination** (schedule to be announced by DigiPen Admin)

Grading Policy

The examination on week fifteen is *optional*. You must inform the instructor of your decision to not take the final exam *by week fourteen*.

The relative weights of homework, quizzes and exams are:

5% Homework (at least ten)
7.5% Laboratory Worksheets (at least ten, given during the laboratory sessions)
27.5% Quizzes (given during the laboratory session on the noted dates)
60% Examinations (drop the lowest)

Grades will be computed out of 40 points. Letter grades will be computed subject to:

35 = at least A
30 = at least B
20 = at least C- (passing)

To pass the course, you need to

have a passing examination average *and* ***the course total should be greater than or equal to 20****.*

Late Policy

Late assignments **will not** be accepted. There will be **no make-up** quizzes or exams, unless authorized by the instructor.

On Use of Calculators

Calculator use is *discouraged* for this course, and *will not be allowed* during examinations. More sophisticated computing devices now regularly dispense as output the details taught in the course, without the benefit of understanding the result. Calculators can be useful for doing away with the tedious clerical nature of computation, but this course will not evaluate students on their arithmetic—for most part, in-examination computations will be allowed to be left unsimplified without penalty.

Last Day to Withdraw

The final date to withdraw from this course is **28 October 2018**. Scores for six (6) homework submissions, six (6) laboratory worksheets, two (2) quizzes and one (1) examination should be available before this date. In order to withdraw from a course, in accordance with policy, contact your advisor or the Registrar to begin the withdrawal process—it is *not sufficient* simply to stop attending class or to inform the instructor. The last day for withdrawal from this course is cited in the official catalog.

Academic Integrity Policy

Academic dishonesty *in any form* will not be tolerated in this course. Cheating, copying, plagiarizing, or any other form of academic dishonesty (including doing someone else's individual assignments) will result in, at the very minimum, a zero on the assignment in question, and could result in a failing grade in the course or even expulsion from DigiPen.

External Preparation

It is expected that the students in this class spend eight (8) hours on average per week for outside classroom activities through the semester, including, but not limited to, homework, reading assignments, project implementation, group discussions, preparation of examinations, etc.

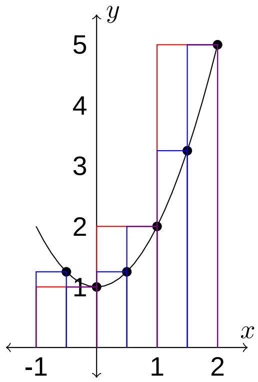
Disability Support Service

Students who have special needs or medical conditions and require formal accommodations in order to fully participate or effectively demonstrate learning in this class should contact the Student Life & Advising Office (`studentlife.sg@digipen.edu`) at the beginning of each semester. A Student Life & Advising Officer will meet with the student privately to discuss how the accommodations will be implemented.

Name: ____________________

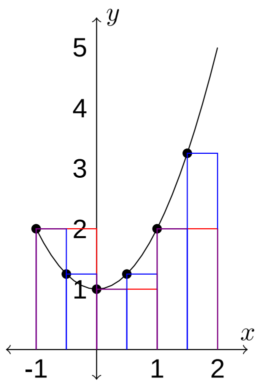
§5.1 #5 (a) Estimate the area under the graph of $f(x) = 1+x^2$ from $x = -1$ to $x = 2$ using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.

For three rectangles, $A \approx 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 5 = 8$. For six rectangles, $A \approx \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{13}{4} + \frac{1}{2} \cdot 5 = \frac{55}{8} = 6.875$.



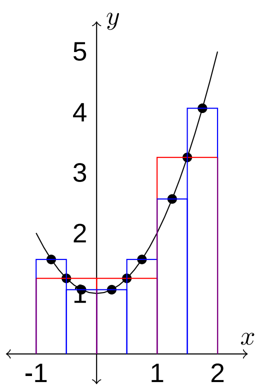
(b) Repeat part (a) using left endpoints.

For three rectangles, $A \approx 1 \cdot 2 + 1 \cdot 1 + 1 \cdot 2 = 5$. For six rectangles, $A \approx \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{13}{4} = \frac{43}{8} = 5.375$.



(c) Repeat part (a) using midpoints.

For three rectangles, $A \approx 1 \cdot \frac{5}{4} + 1 \cdot \frac{5}{4} + 1 \cdot \frac{13}{4} = \frac{23}{4} = 5.75$. For six rectangles, $A \approx \frac{1}{2} \cdot \frac{25}{16} + \frac{1}{2} \cdot \frac{17}{16} + \frac{1}{2} \cdot \frac{17}{16} + \frac{1}{2} \cdot \frac{25}{16} + \frac{1}{2} \cdot \frac{41}{16} + \frac{1}{2} \cdot \frac{65}{16} = \frac{95}{16} = 5.9375$.



(d) From your sketches in parts (a)–(c), which appears to be the best estimate?
As using right endpoints tended to overestimate, and using left endpoints tended to underestimate, the best estimates appear to be those using midpoints. As can be determined, the area is 6 square units.

§5.1 #21 Use

The *area* A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

to find an expression for the area under the graph of $f = \dfrac{2x}{x^2+1}$, $1 \le x \le 3$, as a limit. Do not evaluate the limit.

Using n partitions, $\Delta x = \dfrac{2}{n}$, and $x_i = 1 + \dfrac{2i}{n}$. Thus,

$$A = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} \frac{2+(4i/n)}{[1+(2i/n)]^2+1} = 4\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n+2i}{(n+2i)^2+n^2}.$$

It can be determined later that $A = \ln 5$.

§5.1 #27 Let A be the area under the graph of an increasing continuous function f from a to b, and let L_n and R_n be the approximations to A with n subintervals using left and right endpoints, respectively.

(a) How are A, L_n and R_n related?
Since f is increasing throughout the interval, the left endpoints are minimums over the subintervals, with the right endpoints are maximums over the subintervals, so $L_n < A < R_n$.

(b) Show that $R_n - L_n = \dfrac{b-a}{n}[f(b) - f(a)]$. Then draw a diagram to illustrate this equation by showing that the n rectangles representing $R_n - L_n$ can be reassembled to form a single rectangle whose area is the right side of the equation.

If the partition points are $a = x_0, x_1, \ldots, x_{n-1}, x_n = b$, with $\Delta x = x_i - x_{i-1} = \dfrac{b-a}{n}$, for $1 \le i \le n$, then $R_n = \Delta x[f(x_1) + \cdots + f(x_n)]$ and $L_n = \Delta x[f(x_0) + \cdots + f(x_{n-1})]$, so $R_n - L_n = \Delta x[f(x_n) - f(x_0)] = \dfrac{b-a}{n}[f(b) - f(a)]$.
In the diagram, for each subinterval $[x_{i-1}, x_i]$, the difference between the approximations for the interval is a rectangle whose area is $\Delta x[f(x_i) - f(x_{i-1})]$. If these rectangles are stacked on top of each other, their area would be $\Delta x \sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]$, which telescopes to $\Delta x[f(x_n) - f(x_0)]$—that is, it forms a rectangle whose width is Δx, and whose length is $f(x_n) - f(x_0)$.

(c) Deduce that $R_n - A < \dfrac{b-a}{n}[f(b) - f(a)]$.

Since $L_n < A < R_n$, $R_n - A < R_n - L_n = \dfrac{b-a}{n}[f(b) - f(a)]$.

§5.2 #67 Which of the integrals $\displaystyle\int_1^2 \arctan x\,\mathrm{d}x$, $\displaystyle\int_1^2 \arctan\sqrt{x}\,\mathrm{d}x$, and $\displaystyle\int_1^2 \arctan(\sin x)\,\mathrm{d}x$ has the largest value? Why?

By Property 7, the order of these integrals can be determined by the order of the integrated functions $\arctan x$, $\arctan\sqrt{x}$ and $\arctan(\sin x)$ on the interval $[1, 2]$. $\arctan x$ is increasing everywhere, so the order of the functions can be determined by the order of the composed functions x, $\sqrt{x}$ and $\sin x$ on the interval $[1, 2]$—there, $\sin x \le \sqrt{x} \le x$. Thus, the largest of the values is $\displaystyle\int_1^2 \arctan x\,\mathrm{d}x$.

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

Name: ______________________________

§5.5 #78 Evaluate $\int_0^1 x\sqrt{1-x^4}\,\mathrm{d}x$ by making a substitution and interpreting the resulting integral in terms of an area.

Let $u = x^2$: then $\mathrm{d}u = 2x\,\mathrm{d}x$ and $\int_0^1 x\sqrt{1-x^4}\,\mathrm{d}x = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,\mathrm{d}u$. This is half the area of the portion of the unit circle in the first quadrant, thus $\int_0^1 x\sqrt{1-x^4}\,\mathrm{d}x = \frac{1}{2}\left[\frac{1}{4}\pi\right] = \frac{\pi}{8}$.

§5.5 #94 (a) If f is continuous, prove that $\int_0^{\pi/2} f(\cos x)\,\mathrm{d}x = \int_0^{\pi/2} f(\sin x)\,\mathrm{d}x$.

Using the substitution $u = \frac{\pi}{2} - x$, $x = \frac{\pi}{2} - u$, and $\mathrm{d}u = -\,\mathrm{d}x$, so

$$\int_0^{\pi/2} f(\cos x)\,\mathrm{d}x = \int_{\pi/2}^{0} -f\left[\cos\left(\frac{\pi}{2} - u\right)\right]\,\mathrm{d}u = \int_0^{\pi/2} f(\sin u)\,\mathrm{d}u = \int_0^{\pi/2} f(\sin x)\,\mathrm{d}x.$$

(b) Use part (a) to evaluate $\int_0^{\pi/2} \cos^2 x\,\mathrm{d}x$ and $\int_0^{\pi/2} \sin^2 x\,\mathrm{d}x$.

Since

$$\int_0^{\pi/2} \cos^2 x\,\mathrm{d}x + \int_0^{\pi/2} \sin^2 x\,\mathrm{d}x = \int_0^{\pi/2} (\cos^2 x + \sin^2 x)\,\mathrm{d}x = \int_0^{\pi/2} \mathrm{d}x = \frac{\pi}{2},$$

and $f(x) = x^2$ is continuous, from above, $\int_0^{\pi/2} \cos^2 x\,\mathrm{d}x = \int_0^{\pi/2} \sin^2 x\,\mathrm{d}x = \frac{1}{2}\left[\frac{\pi}{2}\right] = \frac{\pi}{4}$.

§7.1 #32 Evaluate $\int_1^2 \frac{(\ln x)^2}{x^3}\,\mathrm{d}x$.

Let $u = (\ln x)^2$: then $\mathrm{d}u = \frac{2\ln x\,\mathrm{d}x}{x}$, $\mathrm{d}v = \frac{\mathrm{d}x}{x^3}$, so $v = -\frac{1}{2x^2}$ and

$$\int_1^2 \frac{(\ln x)^2}{x^3}\,\mathrm{d}x = \left[-\frac{1}{2}\left(\frac{\ln x}{x}\right)^2\right]_1^2 + \int_1^2 \frac{\ln x}{x^3}\,\mathrm{d}x.$$

Let $U = \ln x$: then $\mathrm{d}U = \frac{\mathrm{d}x}{x}$, $\mathrm{d}V = \frac{\mathrm{d}x}{x^3}$, so $V = -\frac{1}{2x^2}$ and

$$\int_1^2 \frac{(\ln x)^2}{x^3}\,\mathrm{d}x = \left[-\frac{(\ln x)^2 + \ln x}{2x^2}\right]_1^2 + \frac{1}{2}\int_1^2 \frac{\mathrm{d}x}{x^3} = \left[-\frac{2(\ln x)^2 + 2\ln x + 1}{4x^2}\right]_1^2$$
$$= \frac{1}{4} - \frac{2(\ln 2)^2 + 2\ln 2 + 1}{16} = \frac{3 - \ln 4 - 2(\ln 2)^2}{16} \approx 0.0408.$$

§7.1 #70 If $f(0) = g(0) = 0$ and f'' and g'' are continuous, show that

$$\int_0^a f(x)g''(x)\,\mathrm{d}x = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,\mathrm{d}x.$$

Let $u = f(x)$: then $\mathrm{d}u = f'(x)\,\mathrm{d}x$, $\mathrm{d}v = g''(x)\,\mathrm{d}x$, so $v = g'(x)$ and

$$\int_0^a f(x)g''(x)\,\mathrm{d}x = [f(x)g'(x)]_0^a - \int_0^a f'(x)g'(x)\,\mathrm{d}x = f(a)g'(a) - \int_0^a f'(x)g'(x)\,\mathrm{d}x.$$

Likewise,

$$\int_0^a f''(x)g(x)\,\mathrm{d}x = [f'(x)g(x)]_0^a - \int_0^a f'(x)g'(x)\,\mathrm{d}x = f'(a)g(a) - \int_0^a f'(x)g'(x)\,\mathrm{d}x,$$

thus $\int_0^a f'(x)g'(x)\,\mathrm{d}x = f'(a)g(a) - \int_0^a f''(x)g(x)\,\mathrm{d}x$, and the statement follows by substitution.

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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§5.5 #77 Evaluate $\int_{-2}^{2}(x+3)\sqrt{4-x^2}\,\mathrm{d}x$ by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.

$\int_{-2}^{2}(x+3)\sqrt{4-x^2}\,\mathrm{d}x = 3\int_{-2}^{2}\sqrt{4-x^2}\,\mathrm{d}x + \int_{-2}^{2}x\sqrt{4-x^2}\,\mathrm{d}x$. The first integral is thrice the area of the semicircle above the x-axis $x^2+y^2=4$, which is 6π; using $u=4-x^2$ with the second interval gives $\mathrm{d}u=-2x\,\mathrm{d}x$ and $\int_{-2}^{2}x\sqrt{4-x^2}\,\mathrm{d}x = -\frac{1}{2}\int_{0}^{0}\sqrt{u}\,\mathrm{d}u = 0$. Thus, the integral is equal to 6π.

§5.5 #91 If a and b are positive numbers, show that $\int_0^1 x^a(1-x)^b\,\mathrm{d}x = \int_0^1 x^b(1-x)^a\,\mathrm{d}x$.

Using $y=1-x$, gives $\mathrm{d}y=-\,\mathrm{d}x$ and $\int_0^1 x^a(1-x)^b\,\mathrm{d}x = -\int_1^0(1-y)^a y^b\,\mathrm{d}y = \int_0^1 y^b(1-y)^a\,\mathrm{d}y = \int_0^1 x^b(1-x)^a\,\mathrm{d}x$, which results from changing the dummy variable of the integral from y to x.

§7.1 #53 Use integration by parts to prove $\int \tan^n x\,\mathrm{d}x = \dfrac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,\mathrm{d}x$, $(n\neq 1)$.

Noting $\int \tan x\,\mathrm{d}x = \ln|\sec x| + C$, consider for $n\geq 2$, that $\tan^n x = \tan^{n-2}x(\sec^2 x - 1)$, so $\int \tan^n x\,\mathrm{d}x = \int \tan^{n-2}x\sec^2 x\,\mathrm{d}x - \int \tan^{n-2}x\,\mathrm{d}x$. For the first integral, letting $u=\tan x$ sets $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$, such that

$$\int \tan^n x\,\mathrm{d}x = \int u^{n-2}\,\mathrm{d}u - \int \tan^{n-2}x\,\mathrm{d}x = \frac{u^{n-1}}{n-1} - \int \tan^{n-2}x\,\mathrm{d}x = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,\mathrm{d}x.$$

§7.1 #67 The Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right)\mathrm{d}t$ was discussed in Example 5.3.3 and is used extensively in the theory of optics. Find $\int S(x)\,\mathrm{d}x$. [Your answer will involve $S(x)$.]

Let $u=S(x)$: then $\mathrm{d}v=\,\mathrm{d}x$ gives $v=x$ and $\mathrm{d}u=\sin\left(\frac{1}{2}\pi x^2\right)\mathrm{d}x$, so

$$\int S(x)\,\mathrm{d}x = xS(x) - \int x\sin\left(\frac{1}{2}\pi x^2\right)\mathrm{d}x.$$

Let $w=\dfrac{\pi x^2}{2}$: then $\mathrm{d}w=\pi x\,\mathrm{d}x$, so

$$\int S(x)\,\mathrm{d}x = xS(x) - \frac{1}{\pi}\int \sin w\,\mathrm{d}w = xS(x) + \frac{1}{\pi}\cos w + C = xS(x) + \frac{1}{\pi}\cos\left(\frac{1}{2}\pi x^2\right) + C.$$

§7.1 #71 Suppose that $f(1)=2$, $f(4)=7$, $f'(1)=5$, $f'(4)=3$, and f'' is continuous. Find the value of $\int_1^4 xf''(x)\,\mathrm{d}x$.

Letting $u=x$ sets $\mathrm{d}v=f''(x)\,\mathrm{d}x$, such that $\mathrm{d}u=\,\mathrm{d}x$, $v=f'(x)$ and $\int_1^4 xf''(x)\,\mathrm{d}x = xf'(x)]_{x=1}^{x=4} - \int_1^4 f'(x)\,\mathrm{d}x = 4f'(4) - f'(1) - f(4) + f(1) = 12-5-7+2 = 2$.

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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This is an open-book quiz. Work quietly and individually. You may use a calculator, but not any internet-enabled devices. Make sure that there is a seat between you and your neighbors.

Evaluate the following (1 point each):

1. $\displaystyle\int x \sinh x^2 \, \mathrm{d}x$

   Let $u = x^2$: then $\mathrm{d}u = 2x\,\mathrm{d}x$ and $\displaystyle\int x\sinh x^2\,\mathrm{d}x = \frac{1}{2}\int \sinh u\,\mathrm{d}u = \frac{1}{2}\cosh u + C = \frac{1}{2}\cosh x^2 + C$.

   Check: $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\cosh x^2 + C\right] = \frac{1}{2}(\sinh x^2)2x.$

2. $\displaystyle\int \cos^{-1} x \, \mathrm{d}x$

   Let $u = \cos^{-1} x$ and $\mathrm{d}v = \mathrm{d}x$: then $\mathrm{d}u = -\dfrac{\mathrm{d}x}{\sqrt{1-x^2}}$, $v = x$ and $\displaystyle\int \cos^{-1} x\,\mathrm{d}x = x\cos^{-1} x + \int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}$. Let $w = 1 - x^2$: then $\mathrm{d}w = -2x\,\mathrm{d}x$ and $\displaystyle\int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} = -\frac{1}{2}\int\frac{\mathrm{d}w}{\sqrt{w}} = -\sqrt{w} + C$. Thus, $\displaystyle\int \cos^{-1} x\,\mathrm{d}x = x\cos^{-1} x - \sqrt{1-x^2} + C$.

   Check: $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left[x\cos^{-1} x - \sqrt{1-x^2} + C\right] = \cos^{-1} x - \frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} = \cos^{-1} x.$

3. $\displaystyle\int_0^1 \left(t^3 - \frac{2}{3}t\right) e^{3t}\,\mathrm{d}t$

   Let $u = t^3$ and $\mathrm{d}v = e^{3t}\,\mathrm{d}t$: then $\mathrm{d}u = 3t^2\,\mathrm{d}t$, $v = \dfrac{1}{3}e^{3t}$ and $\displaystyle\int\left(t^3 - \frac{2}{3}t\right)e^{3t}\,\mathrm{d}t = \frac{1}{3}t^3e^{3t} - \int\left(t^2 + \frac{2}{3}t\right)e^{3t}\,\mathrm{d}t$. Let $U = t^2$ and $\mathrm{d}V = e^{3t}\,\mathrm{d}t$: then $\mathrm{d}U = 2t\,\mathrm{d}t$, $V = \dfrac{1}{3}e^{3t}\,\mathrm{d}t$ and $\displaystyle\int\left(t^2 + \frac{2}{3}t\right)e^{3t}\,\mathrm{d}t = \frac{1}{3}t^2e^{3t} + C$. Thus, $\displaystyle\int\left(t^3 - \frac{2}{3}t\right)e^{3t}\,\mathrm{d}t = \frac{t^3 - t^2}{3}e^{3t} + C$, and $\displaystyle\int_0^1\left(t^3 - \frac{2}{3}t\right)e^{3t}\,\mathrm{d}t = 0.$

   Check: $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{3}(t^3 - t^2)e^{3t} + C\right] = \frac{1}{3}\left[(3t^2 - 2t)e^{3t} + (3t^3 - 3t^2)e^{3t}\right] = \frac{3t^3 - 2t}{3}e^{3t}.$

4. $\displaystyle\int (\ln x^3)^2\,\mathrm{d}x$

   Note $(\ln x^3)^2 = (3\ln x)^2 = 9(\ln x)^2$. Let $u = (\ln x)^2$ and $\mathrm{d}v = \mathrm{d}x$: then $\mathrm{d}u = \dfrac{2\ln x\,\mathrm{d}x}{x}$, $v = x$ and $\displaystyle\int(\ln x)^2\,\mathrm{d}x = x(\ln x)^2 - 2\int\ln x\,\mathrm{d}x$. Let $U = \ln x$ and $\mathrm{d}V = \mathrm{d}x$: then $\mathrm{d}U = \dfrac{\mathrm{d}x}{x}$, $V = x$ and $\displaystyle\int\ln x\,\mathrm{d}x = x\ln x - \int\mathrm{d}x$. Thus, $\displaystyle\int(\ln x^3)^2\,\mathrm{d}x = 9x(\ln x)^2 - 18x\ln x + 18x + C$.

   Check: $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left[9x(\ln x)^2 - 18x\ln x + 18x + C\right] = 9(\ln x)^2 + 18\ln x - 18\ln x - 18 + 18 = 9(\ln x)^2 = (\ln x^3)^2.$

5. $\displaystyle\int x\cosh x\,\mathrm{d}x$

Let $u = x$ and $\mathrm{d}v = \cosh x\ \mathrm{d}x$: then $\mathrm{d}u = \ \mathrm{d}x$, $v = \sinh x$ and $\displaystyle\int x\cosh x\ \mathrm{d}x = x\sinh x - \int \sinh x\ \mathrm{d}x = x\sinh x - \cosh x + C$.
Check: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[x\sinh x - \cosh x + C\right] = \sinh x + x\cosh x - \sinh x = x\cosh x$.

6. $\displaystyle\int_0^{\sin(1/2)} \sqrt{\frac{1-4(\sin^{-1}x)^2}{1-x^2}}\ \mathrm{d}x$

Let $u = \sin^{-1}x$: then $\mathrm{d}u = \dfrac{\mathrm{d}x}{\sqrt{1-x^2}}$ and $\displaystyle\int_0^{\sin(1/2)} \sqrt{\frac{1-4(\sin^{-1}x)^2}{1-x^2}}\ \mathrm{d}x = \int_0^{1/2}\sqrt{1-4u^2}\ \mathrm{d}u = 2\int_0^{1/2}\sqrt{\frac{1}{4}-x^2}\ \mathrm{d}x$. This is twice the area of the portion of the circle $x^2+y^2 = \dfrac{1}{4}$ in the first quadrant, thus $\displaystyle\int_0^{\sin(1/2)} \sqrt{\frac{1-4(\sin^{-1}x)^2}{1-x^2}}\ \mathrm{d}x = \frac{2}{4}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{8}$.

$$\mathit{Check}: \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{4}\sin^{-1}(2\sin^{-1}x) + \frac{1}{2}(\sin^{-1}x)\sqrt{1-4(\sin^{-1}x)^2} + C\right]$$
$$= \frac{1}{2\sqrt{[1-4(\ln x)^2](1-x^2)}} + \frac{1}{2}\sqrt{\frac{1-4(\sin^{-1}x)^2}{1-x^2}} - \frac{2(\sin^{-1}x)^2}{\sqrt{[1-4(\sin^{-1}x)^2](1-x^2)}}$$
$$= \frac{1-4(\sin^{-1}x)^2}{2\sqrt{[1-4(\sin^{-1}x)^2](1-x^2)}} + \frac{1}{2}\sqrt{\frac{1-4(\sin^{-1}x)^2}{1-x^2}} = \sqrt{\frac{1-4(\sin^{-1}x)^2}{1-x^2}}$$

$$\left[\frac{1}{4}\sin^{-1}(2\sin^{-1}x) + \frac{1}{2}(\sin^{-1}x)\sqrt{1-4(\sin^{-1}x)^2}\right]_0^{\sin(1/2)} = \frac{1}{4}\sin^{-1}(1) + \frac{1}{2}\left(\frac{1}{2}\right)\sqrt{1-4\left(\frac{1}{2}\right)^2} = \frac{\pi}{8}.$$

7. $\displaystyle\int \sin 2x\exp(\sin^2 x)\ \mathrm{d}x$

Let $u = \sin^2 x$: then $\mathrm{d}u = 2\sin x\cos x\ \mathrm{d}x = \sin 2x\ \mathrm{d}x$ and $\displaystyle\int \sin 2x\exp(\sin^2 x)\ \mathrm{d}x = \int e^u\ \mathrm{d}u = e^u + C = \exp(\sin^2 x) + C$.
Check: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\exp(\sin^2 x) + C\right] = 2\sin x\cos x\exp(\sin^2 x) = \sin 2x\exp(\sin^2 x)$.

8. $\displaystyle\int \sinh 2x\sin x\ \mathrm{d}x$

Let $u = \sinh 2x$ and $\mathrm{d}v = \sin x\ \mathrm{d}x$: then $\mathrm{d}u = 2\cosh 2x\ \mathrm{d}x$, $v = -\cos x$ and $\displaystyle\int \sinh 2x\sin x\ \mathrm{d}x = -\sinh 2x\cos x + 2\int\cosh 2x\cos x\ \mathrm{d}x$. Let $U = \cosh 2x$ and $\mathrm{d}V = \cos x\ \mathrm{d}x$: then $\mathrm{d}U = 2\sinh 2x\ \mathrm{d}x$, $V = \sin x$ and $\displaystyle\int\cosh 2x\cos x\ \mathrm{d}x = \cosh 2x\sin x - 2\int\sinh 2x\sin x\ \mathrm{d}x$. Thus,

$$\int \sinh 2x\sin x\ \mathrm{d}x = -\sinh 2x\cos x + 2\cosh 2x\sin x - 4\int\sinh 2x\sin x\ \mathrm{d}x,$$
$$5\int \sinh 2x\sin x\ \mathrm{d}x = 2\cosh 2x\sin x - \cos x\sinh 2x + C,$$
$$\int \sinh 2x\sin x\ \mathrm{d}x = \frac{2}{5}\cosh 2x\sin x - \frac{1}{5}\cos x\sinh 2x + C.$$

Check: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{2}{5}\cosh 2x\sin x - \dfrac{1}{5}\cos x\sinh 2x + C\right] = \dfrac{2}{5}\cosh 2x\cos x + \dfrac{4}{5}\sinh 2x\sin x + \dfrac{1}{5}\sin x\sinh 2x - \dfrac{2}{5}\cos x\cosh 2x = \sinh 2x\sin x$.

Find $\displaystyle\int \sin^m x \cos^n x \, \mathrm{d}x$, $m, n \geq 0$, integers.

- If m is odd
  1. Replace $\sin^{m-1} x = (1 - \cos^2 x)^{(m-1)/2}$:
  $$\int \sin^m x \cos^n x \, \mathrm{d}x = \int (1 - \cos^2 x)^{(m-1)/2} \cos^n x \sin x \, \mathrm{d}x = \int p(\cos x) \sin x \, \mathrm{d}x,$$
  where p is a polynomial.
  2. Use the substitution $u = \cos x$ and integrate: $\mathrm{d}u = -\sin x \, \mathrm{d}x$ and
  $$\int \sin^m x \cos^n x \, \mathrm{d}x = -\int p(u) \, \mathrm{d}u = -P(u) + C = -P(\cos x) + C,$$
  where P is a polynomial such that $P' = p$.
- If n is odd
  1. Replace $\cos^{n-1} x = (1 - \sin^2 x)^{(n-1)/2}$:
  $$\int \sin^m x \cos^n x \, \mathrm{d}x = \int \sin^m x (1 - \sin^2 x)^{(n-1)/2} \cos x \, \mathrm{d}x = \int p(\sin x) \cos x \, \mathrm{d}x,$$
  where p is a polynomial.
  2. Use the substitution $u = \sin x$ and integrate: $\mathrm{d}u = \cos x \, \mathrm{d}x$ and
  $$\int \sin^m x \cos^n x \, \mathrm{d}x = \int p(u) \, \mathrm{d}u = P(u) + C = P(\sin x) + C,$$
  where P is a polynomial such that $P' = p$.
- Otherwise (m and n are both even)

  1a. If $m \geq n$: replace $\sin^n x \cos^n x = \left(\dfrac{\sin 2x}{2}\right)^n$ and $\sin^{m-n} x = \left(\dfrac{1 - \cos 2x}{2}\right)^{(m-n)/2}$
  $$\int \sin^m x \cos^n x \, \mathrm{d}x = \frac{1}{2^{(m+n)/2}} \int \sin^n 2x (1 - \cos 2x)^{(m-n)/2} \, \mathrm{d}x.$$

  1b. If $m < n$: replace $\sin^m x \cos^m x = \left(\dfrac{\sin 2x}{2}\right)^m$ and $\cos^{n-m} x = \left(\dfrac{1 + \cos 2x}{2}\right)^{(n-m)/2}$
  $$\int \sin^m x \cos^n x \, \mathrm{d}x = \frac{1}{2^{(m+n)/2}} \int \sin^m 2x (1 + \cos 2x)^{(n-m)/2} \, \mathrm{d}x.$$

  2. Use substitution $u = 2x$ and perform termwise integration (`return to top`): $\mathrm{d}u = 2 \, \mathrm{d}x$,
  $$\int \sin^m x \cos^n x \, \mathrm{d}x = \frac{1}{2^{(m+n+2)/2}} \sum_{k=0}^{|n-m|/2} \operatorname{sign}(n-m)^k \binom{|n-m|/2}{k} \int \sin^{\min(n,m)} u \cos^k u \, \mathrm{d}u.$$
  Since the power of $\sin x$ will still be even, terms where k is odd can be solved by the second case above, but this case will be revisited when k is even.

Find $\displaystyle\int \frac{\sin^m x}{\cos^n x} \, \mathrm{d}x$, $m, n \geq 0$, integers. (Analogous process for $\displaystyle\int \frac{\cos^m x}{\sin^n x} \, \mathrm{d}x$, using cofunctions.)

- If $m > n$

1. Replace $\sin^m x = (1-\cos^2 x)^{\lfloor m/2 \rfloor} \sin^{m \mod 2} x$
$$\int \frac{\sin^m x}{\cos^n x}\,\mathrm{d}x = \int \frac{(1-\cos^2 x)^{\lfloor m/2 \rfloor} \sin^{m \mod 2} x}{\cos^n x}\,\mathrm{d}x.$$
2. Perform termwise integration (`go to next cases` or `use previous algorithm`)
$$\int \frac{\sin^m x}{\cos^n x}\,\mathrm{d}x = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \binom{\lfloor m/2 \rfloor}{k} \int \sin^{m \mod 2} x \cos^{2k-n} x\,\mathrm{d}x.$$
If $2k-n \geq 0$, the previous algorithm is used, otherwise the next cases are used.

- If $m=n$, $\dfrac{\sin^n x}{\cos^n x} = \tan^n x$: Use $\displaystyle\int \tan x\,\mathrm{d}x = \int \frac{\sin x\,\mathrm{d}x}{\cos x} = -\int \frac{\mathrm{d}(\cos x)}{\cos x} = \ln|\sec x| + C$, and,
$$\int \tan^n x\,\mathrm{d}x = \int \tan^{n-2} x \sec^2 x\,\mathrm{d}x - \int \tan^{n-2} x\,\mathrm{d}x = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,\mathrm{d}x, \text{ for } n > 1.$$
- Otherwise $\left(n > m, \dfrac{\sin^m x}{\cos^n x} = \tan^m x \sec^{n-m} x\right)$
  - If $n-m$ is even
    1. Replace $\sec^{n-m-2} x = (1+\tan^2 x)^{(n-m-2)/2}$:
$$\int \frac{\sin^m x}{\cos^n x}\,\mathrm{d}x = \int \tan^m x (1+\tan^2 x)^{(n-m-2)/2} \sec^2 x\,\mathrm{d}x = \int p(\tan x)\sec^2 x\,\mathrm{d}x,$$
where p is a polynomial.
    2. Use the substitution $u = \tan x$ and integrate: $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$ and
$$\int \frac{\sin^m x}{\cos^n x}\,\mathrm{d}x = \int p(u)\,\mathrm{d}u = P(u) + C = P(\tan x) + C,$$
where P is a polynomial such that $P' = p$.
  - If m is odd
    1. Replace $\tan^{m-1} x = (\sec^2 x - 1)^{(m-1)/2}$:
$$\int \frac{\sin^m x}{\cos^n x}\,\mathrm{d}x = \int (\sec^2 x - 1)^{(m-1)/2} \sec^{n-m} x \tan x\,\mathrm{d}x = \int p(\sec x)\sec x \tan x\,\mathrm{d}x,$$
where p is a polynomial.
    2. Use the substitution $u = \sec x$ and integrate: $\mathrm{d}u = \sec x \tan x\,\mathrm{d}x$ and
$$\int \frac{\sin^m x}{\cos^n x}\,\mathrm{d}x = \int p(u)\,\mathrm{d}u = P(u) + C = P(\sec x) + C,$$
where P is a polynomial such that $P' = p$.
  - Otherwise (m is even and n is odd)
    1. Replace $\tan^m x = (\sec^2 x - 1)^{m/2}$:
$$\int \frac{\sin^m x}{\cos^n x}\,\mathrm{d}x = \int (\sec^2 x - 1)^{m/2} \sec^{n-m} x\,\mathrm{d}x = \sum_{k=0}^{m/2} (-1)^k \binom{m/2}{k} \int \sec^{n-2k} x\,\mathrm{d}x.$$
    2. Use $\displaystyle\int \sec x\,\mathrm{d}x = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,\mathrm{d}x = \int \frac{\mathrm{d}(\sec x + \tan x)}{\sec x + \tan x} = \ln|\sec x + \tan x| + C$,
and, for $n > 0$, by integration by parts ($u = \sec^{2n-1} x$ and $\mathrm{d}v = \sec^2 x\,\mathrm{d}x$):
$$\begin{aligned}\int \sec^{2n+1} x\,\mathrm{d}x &= \sec^{2n-1} x \tan x - (2n-1)\int \sec^{2n-1} x \tan^2 x\,\mathrm{d}x \\ &= \sec^{2n-1} x \tan x - (2n-1)\int \sec^{2n+1} x\,\mathrm{d}x + (2n-1)\int \sec^{2n-1} x\,\mathrm{d}x \\ &= \frac{1}{2n}\sec^{2n-1} x \tan x + \frac{2n-1}{2n}\int \sec^{2n-1} x\,\mathrm{d}x.\end{aligned}$$

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§7.2 #56 Evaluate $\int \sin x \cos x \, \mathrm{d}x$ by four methods:

(a) the substitution $u = \cos x$

This gives $\mathrm{d}u = -\sin x \, \mathrm{d}x$, thus $\int \sin x \cos x \, \mathrm{d}x = -\int u \, \mathrm{d}u = -\frac{u^2}{2} + C = -\frac{\cos^2 x}{2} + C$.

(b) the substitution $u = \sin x$

This gives $\mathrm{d}u = \cos x \, \mathrm{d}x$, thus $\int \sin x \cos x \, \mathrm{d}x = \int u \, \mathrm{d}u = \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C$.

(c) the identity $\sin 2x = 2 \sin x \cos x$

Letting $u = 2x$ gives $\mathrm{d}u = 2 \, \mathrm{d}x$, thus $\int \sin x \cos x \, \mathrm{d}x = \frac{1}{4} \int \sin u \, \mathrm{d}u = -\frac{\cos u}{4} + C = -\frac{\cos 2x}{4} + C$.

(d) integration by parts

Letting $u = \cos x$ sets $\mathrm{d}v = \sin x \, \mathrm{d}x$, thus $\mathrm{d}u = -\sin x \, \mathrm{d}x$, $v = -\cos x$ and $\int \sin x \cos x \, \mathrm{d}x = -\cos^2 x - \int \sin x \cos x \, \mathrm{d}x$, so $\int \sin x \cos x \, \mathrm{d}x = -\frac{\cos^2 x}{2} + C$.

Letting $u = \sin x$ sets $\mathrm{d}v = \cos x \, \mathrm{d}x$, thus $\mathrm{d}u = \cos x \, \mathrm{d}x$, $v = \sin x$ and $\int \sin x \cos x \, \mathrm{d}x = \sin^2 x - \int \sin x \cos x \, \mathrm{d}x$, so $\int \sin x \cos x \, \mathrm{d}x = \frac{\sin^2 x}{2} + C$.

Explain the different appearances of the answers.

From the identities $\cos 2x = \cos^2 x - \sin^2 x$ and $\sin^2 x + \cos^2 x = 1$, $-\cos 2x = 1 - 2\cos^2 x = 2\sin^2 x - 1$, so the arbitrary constant C absorbs the constant $\pm\frac{1}{4}$ in the equivalent values for $-\frac{\cos 2x}{4}$, and the expressions are equivalent.

§7.2 #70 A *finite Fourier sine series* is given by the sum $f(x) = \sum_{n=1}^{N} a_n \sin nx = a_1 \sin x + a_2 \sin 2x + \cdots + a_N \sin Nx$. Show that the mth coefficient a_m is given by the formula $a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, \mathrm{d}x$.

First, it must be shown that $\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n, \end{cases}$ where m and n are positive integers: $2 \sin mx \sin nx = \cos[(m-n)x] - \cos[(m+n)x]$, so, if $m \neq n$,

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x = \frac{1}{2} \int_{-\pi}^{\pi} (\cos[(m-n)x] - \cos[(m+n)x]) \, \mathrm{d}x = \left[\frac{\sin[(m-n)x]}{2(m-n)} - \frac{\sin[(m+n)x]}{2(m+n)} \right]_{-\pi}^{\pi},$$

all of whose terms are zero. But, if $m = n$,

$$\int_{-\pi}^{\pi} \sin^2 nx \, \mathrm{d}x = \frac{1}{2} \int_{-\pi}^{\pi} \mathrm{d}x - \frac{1}{2} \int_{-\pi}^{\pi} \cos 2nx \, \mathrm{d}x = \left[\frac{x}{2} - \frac{\sin 2nx}{4n} \right]_{-\pi}^{\pi} = \pi.$$

Then,

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin mx \, \mathrm{d}x &= \int_{-\pi}^{\pi} \left(\sum_{n=1}^{N} a_n \sin nx \sin mx \right) \mathrm{d}x = \sum_{n=1}^{N} \left[a_n \int_{-\pi}^{\pi} \sin nx \sin mx \, \mathrm{d}x \right] \\ &= 0 + \cdots + 0 + a_m \int_{-\pi}^{\pi} \sin mx \sin mx \, \mathrm{d}x + 0 + \cdots = a_m \pi, \end{aligned}$$

and the conclusion follows from dividing both sides by π.

§7.3 #32 Evaluate $\displaystyle\int \frac{x^2}{(x^2+a^2)^{3/2}}\,\mathrm{d}x$

(a) by trigonometric substitution
Using $x = a\tan\theta$ gives $\mathrm{d}x = a\sec^2\theta\,\mathrm{d}\theta$, $\sqrt{x^2+a^2} = a\sec\theta$ and

$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\,\mathrm{d}x = \int \frac{a^2\tan^2\theta(a\sec^2\theta\,\mathrm{d}\theta)}{a^3\sec^3\theta} = \int \frac{\sin^2\theta\,\mathrm{d}\theta}{\cos\theta} = \int \frac{(1-\cos^2\theta)\,\mathrm{d}\theta}{\cos\theta}$$
$$= \int \sec\theta\,\mathrm{d}\theta - \int \cos\theta\,\mathrm{d}\theta = \ln|\sec\theta+\tan\theta| - \sin\theta + C$$
$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| - \frac{x/a}{\sqrt{x^2+a^2}/a} + C = \ln\left|x+\sqrt{x^2+a^2}\right| - \frac{x}{\sqrt{x^2+a^2}} + C.$$

(b) by the hyperbolic substitution $x = a\sinh t$
Using $x = a\sinh t$ gives $\mathrm{d}x = a\cosh t\,\mathrm{d}t$, $\sqrt{x^2+a^2} = a\cosh t$ and

$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\,\mathrm{d}x = \int \frac{a^2\sinh^2 t(a\cosh t\,\mathrm{d}t)}{a^3\cosh^3 t} = \int \tanh^2 t\,\mathrm{d}t = \int (1-\mathrm{sech}^2 t)\,\mathrm{d}t,$$

since dividing both sides of the identity $\cosh^2 t - \sinh^t = 1$ by $\cosh^2 t$ gives $1-\tanh^2 t = \mathrm{sech}^2 t$,

$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\,\mathrm{d}x = \int \mathrm{d}t - \int \mathrm{sech}^2 t\,\mathrm{d}t = t - \tanh t + C = \sinh^{-1}\left(\frac{x}{a}\right) - \frac{x/a}{\sqrt{x^2+a^2}/a} + C,$$

since $\dfrac{\mathrm{d}}{\mathrm{d}t}\tanh t = \dfrac{\cosh^2 t - \sinh^2 t}{\cosh^2 t} = \mathrm{sech}^2 t$, and

$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\,\mathrm{d}x == \ln\left|\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2+1}\right| - \frac{x}{\sqrt{x^2+a^2}} + C = \ln\left|x+\sqrt{x^2+a^2}\right| - \frac{x}{\sqrt{x^2+a^2}} + C,$$

since $y = \sinh^{-1} x$ means $x = \sinh y = \dfrac{e^y - e^{-y}}{2}$ and $e^{2y} - 2xe^y - 1 = 0$: by the quadratic formula, $e^y = \dfrac{2x}{2} \pm \dfrac{\sqrt{(-2x)^2+4}}{4} = x + \sqrt{x^2+1} \geq 0$, thus $y = \ln|x+\sqrt{x^2+1}| = \sinh^{-1} x$.

§7.3 #44 A water storage tank has the shape of a cylinder with diameter 10 m. It is mounted so that the circular cross-sections are vertical. If the depth of the water is 7 m, what percentage of the total capacity is being used?

The percentage of the total capacity occupied by the water in the tank is the same as the area of any vertical circular cross-section that is in water: if such a cross-section has a cartesian grid with units of length one meter, the origin in the center of the circle, and whose positive x-axis is pointing downward, the percentage to be determined is the area of the portion of the circle $x^2+y^2 = 25$ to the right of the line $x = -2$. By symmetry with respect to the x-axis, the percentage is the area under the curve $y = \sqrt{25-x^2}$ from $x = -2$ to $x = 5$, divided by the area of the semicircle, which is $\dfrac{25\pi}{2}$. Thus, using $x = 5\sin\theta$, giving $\mathrm{d}x = 5\cos\theta\,\mathrm{d}\theta$ and $\sqrt{25-x^2} = 5\cos\theta$, the percentage is

$$\frac{2}{25\pi}\int_{-2}^{5}\sqrt{25-x^2}\,\mathrm{d}x = \frac{2}{\pi}\int_{\sin^{-1}(-2/5)}^{\pi/2}\cos^2\theta\,\mathrm{d}\theta = \frac{1}{\pi}\int_{\sin^{-1}(-2/5)}^{\pi/2}\mathrm{d}\theta + \frac{1}{\pi}\int_{\sin^{-1}(-2/5)}^{\pi/2}\cos 2\theta\,\mathrm{d}\theta$$
$$= \frac{1}{2\pi}\left[2\theta + \sin 2\theta\right]_{\sin^{-1}(-2/5)}^{\pi/2} = \frac{1}{2} - \frac{1}{\pi}\sin^{-1}\left(-\frac{2}{5}\right) + 0 - \frac{1}{2\pi}\sin\left[2\sin^{-1}\left(-\frac{2}{5}\right)\right],$$

where $\cos\left[\sin\left(-\dfrac{2}{5}\right)\right] = \dfrac{\sqrt{21}}{5}$ gives the last term as $2\left(-\dfrac{2}{5}\right)\dfrac{\sqrt{21}}{5} = \dfrac{-4\sqrt{21}}{25}$; thus the percentage is $\dfrac{25\pi+4\sqrt{21}}{50\pi} - \dfrac{1}{\pi}\sin^{-1}\left(-\dfrac{2}{5}\right) \approx 0.7476842$ or about 74.8%.

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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§7.2 #65 A particle moves on a straight line with velocity function $v(t) = \sin \omega t \cos^2 \omega t$. Find its position function $s = f(t)$ if $f(0) = 0$.

Since $s(t) = \int_0^t v(x)\,\mathrm{d}x$, as $s(0) = 0$, using $u = \cos \omega x$ gives $\mathrm{d}u = -\omega \sin \omega x\,\mathrm{d}x$ and

$$s(t) = \int_0^t \sin \omega x \cos^2 \omega x\,\mathrm{d}x = -\frac{1}{\omega}\int_1^{\cos \omega t} u^2\,\mathrm{d}u = \left[\frac{u^3}{3\omega}\right]_{\cos \omega t}^1 = \frac{1-\cos^3 \omega t}{3\omega},$$

assuming $\omega \neq 0$, otherwise $v(t) = s(t) \equiv 0$.

§7.3 #13 Evaluate $\int \frac{\sqrt{x^2-9}}{x^3}\,\mathrm{d}x$.

Using $x = 3\sec\theta$ gives $\mathrm{d}x = 3\sec\theta\tan\theta\,\mathrm{d}\theta$ and $\sqrt{x^2-9} = 3\tan\theta$ and, since $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$,

$$\begin{aligned}\int \frac{\sqrt{x^2-9}}{x^3}\,\mathrm{d}x &= \int \frac{3\tan\theta(3\sec\theta\tan\theta\,\mathrm{d}\theta)}{(3\sec\theta)^3}\,\mathrm{d}x = \frac{1}{3}\int \sin^2\theta\,\mathrm{d}\theta = \frac{1}{6}\int \mathrm{d}\theta - \frac{1}{6}\int \cos 2\theta\,\mathrm{d}\theta \\ &= \frac{\theta}{6} - \frac{\sin 2\theta}{12} + C = \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C.\end{aligned}$$

Check: $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C\right] = \frac{1}{6x\sqrt{(x/3)^2-1}} - \frac{2x^2(x/\sqrt{x^2-9}) - 4x\sqrt{x^2-9}}{4x^4} = \frac{1}{2x\sqrt{x^2-9}} - \frac{x^3-2x(x^2-9)}{2x^4\sqrt{x^2-9}} = \frac{x^3-x^3+2x^3-18x}{2x^4\sqrt{x^2-9}} = \frac{x^2-9}{x^3\sqrt{x^2-9}} = \frac{\sqrt{x^2-9}}{x^3}$.

§7.3 #29 Evaluate $\int x\sqrt{1-x^4}\,\mathrm{d}x$.

Letting $u = x^2 = \sin\theta$ gives $\mathrm{d}u = 2x\,\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$, $\sqrt{1-u^2} = \sqrt{1-x^4} = \cos\theta$ and $\int x\sqrt{1-x^4}\,\mathrm{d}x = \frac{1}{2}\int \cos^2\theta\,\mathrm{d}\theta = \frac{1}{4}\int \mathrm{d}\theta + \frac{1}{4}\int \cos 2\theta\,\mathrm{d}\theta = \frac{\theta}{4} + \frac{\sin 2\theta}{8} + C = \frac{1}{4}\sin^{-1}(x^2) + \frac{x^2\sqrt{1-x^4}}{4} + C$.

Check: $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{4}\sin^{-1}(x^2) + \frac{x^2\sqrt{1-x^4}}{4} + C\right] = \frac{x}{2\sqrt{1-x^4}} + \frac{x\sqrt{1-x^4}}{2} - \frac{x^5}{2\sqrt{1-x^4}} = \frac{x + x(1-x^4) - x^5}{2\sqrt{1-x^4}} = \frac{x(1-x^4)}{\sqrt{1-x^4}} = x\sqrt{1-x^4}$.

§7.3 #31 (a) Use trigonometric substitution to show that $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) + C$.

Using $x = a\tan\theta$ gives $\mathrm{d}x = a\sec^2\theta\,\mathrm{d}\theta$, $\sqrt{x^2+a^2} = a\sec\theta$ and $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta\,\mathrm{d}\theta}{a\sec\theta} = \int \sec\theta\,\mathrm{d}\theta = \ln|\sec\theta+\tan\theta| + C = \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C = \ln(x+\sqrt{x^2+a^2}) + C$.

(b) Use the hyperbolic substitution $x = a\sinh t$ to show that $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$.

Using $x = a\sinh t$ gives $\mathrm{d}x = a\cosh t\,\mathrm{d}t$, $\sqrt{x^2+a^2} = a\cosh t$ and $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \int \frac{a\cosh t\,\mathrm{d}t}{a\cosh t} = \int \mathrm{d}t = t + C = \sinh^{-1}\left(\frac{x}{a}\right) + C$.

These formulas are connected by $\sinh^{-1} x = \ln(x+\sqrt{x^2+1})$, $x \in \mathbb{R}$.

Note $\sinh^{-1}\left(\frac{x}{a}\right) = \ln\left(\frac{x}{a} + \sqrt{\left[\frac{x}{a}\right]^2 + 1}\right) = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right|$.

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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Evaluate the integral.

§7.5 #26 $\displaystyle\int_0^1 \frac{(3x^2+1)\,\mathrm{d}x}{x^3+x^2+x+1}$

Since the denominator can be factored into $(x^2+1)(x+1)$, the integrand can be split into

$$\int_0^1 \frac{3x^2+1}{x^3+x^2+x+1}\,\mathrm{d}x = \int_0^1 \frac{A\,\mathrm{d}x}{x+1} + \int_0^1 \frac{Bx+C}{x^2+1}\,\mathrm{d}x = \left[A\ln|x+1| + \frac{B}{2}\ln(x^2+1) + C\tan^{-1}x\right]_0^1$$
$$= \frac{2A+B}{B}\ln 2 + \frac{C\pi}{4},\ \text{where } 3x^2+1 = A(x^2+1)+(Bx+C)(x+1).$$

The unknowns A, B and C are defined by a system: $A+B=3$, $B+C=0$ and $A+C=1$, so $A=2$, $B=1$ and $C=-1$, so $\displaystyle\int_0^1 \frac{(3x^2+1)\,\mathrm{d}x}{x^3+x^2+x+1} = \frac{5}{2}\ln 2 - \frac{\pi}{4}$.

Check: $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left((x+1)^2\sqrt{x^2+1}\right) - \tan^{-1}x\right] = \frac{2(x+1)\sqrt{x^2+1} + \dfrac{x(x+1)^2}{\sqrt{x^2+1}}}{(x+1)^2\sqrt{x^2+1}} - \frac{1}{x^2+1}$$
$$= \frac{2(x^2+1)+x(x+1)-(x+1)}{(x+1)(x^2+1)} = \frac{2x^2+2+x^2+x-x-1}{(x+1)(x^2+1)} = \frac{3x^2+1}{x^3+x^2+x+1}$$

§7.5 #52 $\displaystyle\int \frac{\mathrm{d}x}{x(x^4+1)}$

Since the denominator can be factored into $x(x^2+\sqrt{2}+1)(x^2-\sqrt{2}+1)$, the integrand can be split into

$$\int \frac{\mathrm{d}x}{x(x^4+1)}\,\mathrm{d}x = \int \frac{A\,\mathrm{d}x}{x} + \int \frac{Bx+C}{[x+(1/\sqrt{2})]^2+(1/2)}\,\mathrm{d}x + \int \frac{Dx+E}{[x-(1/\sqrt{2})]^2+(1/2)}\,\mathrm{d}x$$
$$= A\ln|x| + \int \frac{Bu_+ - (B/\sqrt{2}) + C}{u_+^2+(1/\sqrt{2})^2}\,\mathrm{d}x + \int \frac{Du_- + (D/\sqrt{2}) + E}{u_-^2+(1/\sqrt{2})^2}\,\mathrm{d}x$$
$$= A\ln|x| + \frac{B}{2}\ln(x^2+\sqrt{2}x+1) + (\sqrt{2}C-B)\tan^{-1}(\sqrt{2}x+1) + \frac{D}{2}\ln(x^2-\sqrt{2}x+1)$$
$$+ (\sqrt{2}E+D)\tan^{-1}(\sqrt{2}x-1),\ \text{where } u_\pm = x \pm \frac{1}{\sqrt{2}},\ \text{and the unknowns satisfy}$$
$$1 = A(x^4+1) + (Bx+C)x(x^2-\sqrt{2}x+1) + (Dx+E)x(x^2+\sqrt{2}x+1).$$

The unknowns A, B, C, D and E are defined by a system: $A+B+D=0$, $-\sqrt{2}B+C+\sqrt{2}D+E=0$, $B-\sqrt{2}C+D+\sqrt{2}E=0$, $C+E=0$ and $A=1$, so $B=-\dfrac{1}{2}$, $C=-\dfrac{1}{2\sqrt{2}}$, $D=-\dfrac{1}{2}$ and $E=\dfrac{1}{2\sqrt{2}}$, so $\displaystyle\int \frac{\mathrm{d}x}{x(x^4+1)} = \ln|x| - \frac{1}{4}\ln(x^4+1) + C$.

Check: $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left|\frac{x}{\sqrt[4]{x^4+1}}\right|\right] = \frac{\sqrt[4]{x^4+1}}{x}\,\frac{\sqrt[4]{x^4+1} - \dfrac{x^4}{\sqrt[4]{(x^4+1)^3}}}{\sqrt{x^4+1}} = \frac{x^4+1-x^4}{x(x^4+1)} = \frac{1}{x(x^4+1)}$$

§7.5 #56 $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x}+x\sqrt{x}}$

Using $u^2=x$ gives $2u\,\mathrm{d}u = \mathrm{d}x$ and $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x}+x\sqrt{x}} = \int \frac{2u\,\mathrm{d}u}{u+u^3} = 2\tan^{-1}u + 1 = 2\tan^{-1}\sqrt{x} + C$.

Check: $$\frac{\mathrm{d}}{\mathrm{d}x}\left[2\tan^{-1}\sqrt{x}\right] = \frac{2}{x+1}\,\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}+x\sqrt{x}}$$

§7.5 #72 $\displaystyle\int \frac{\ln(x+1)\,\mathrm{d}x}{x^2}$

Using $u = \ln(x+1)$ leaves $\mathrm{d}v = \dfrac{\mathrm{d}x}{x^2}$, thus $\mathrm{d}u = \dfrac{\mathrm{d}x}{x+1}$ and $v = -\dfrac{1}{x}$, and

$$\int \frac{\ln(x+1)\,\mathrm{d}x}{x^2} = -\frac{\ln(x+1)}{x} + \int \frac{\mathrm{d}x}{x(x+1)} = -\frac{\ln(x+1)}{x} + \int \frac{\mathrm{d}x}{x} - \int \frac{\mathrm{d}x}{x+1} = \ln\left|\frac{x}{x+1}\right| - \frac{\ln(x+1)}{x} + C.$$

Check: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left|\dfrac{x}{x+1}\right| - \dfrac{\ln(x+1)}{x}\right] = \dfrac{x+1}{x}\dfrac{1}{(x+1)^2} - \dfrac{\dfrac{x}{x+1} - \ln(x+1)}{x^2} = \dfrac{1}{x(x+1)} - \dfrac{1}{x(x+1)} + \dfrac{\ln(x+1)}{x^2}$

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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This is an open-book quiz. Work quietly and individually. You may use a calculator, but not any internet-enabled devices. Make sure that there is a seat between you and your neighbors.

Evaluate the following (2 points each):

1. $\displaystyle\int \frac{1+\sqrt{(x-1)^3}}{x^2(x-1)}\,\mathrm{d}x$

$\displaystyle\int \frac{1+\sqrt{(x-1)^3}}{x^2(x-1)}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{x^2(x-1)} + \int \frac{\sqrt{x-1}}{x^2}\,\mathrm{d}x$. For the first integral:

$$\int \frac{\mathrm{d}x}{x^2(x-1)} = A\int \frac{\mathrm{d}x}{x} + B\int \frac{\mathrm{d}x}{x^2} + C\int \frac{\mathrm{d}x}{x-1} = A\ln|x| - \frac{B}{x} + C\ln|x-1| + D,$$

where $1 = Ax(x-1) + B(x-1) + Cx^2 = (A+C)x^2 + (B-A)x - B$, so $B = -1 = A$, $C = 1$. For the second integral: using $u = \sqrt{x-1}$ gives $\mathrm{d}u = \dfrac{\mathrm{d}x}{2\sqrt{x-1}}$, $x = u^2+1$ and

$$\int \frac{\sqrt{x-1}}{x^2}\,\mathrm{d}x = 2\int \frac{(\sqrt{x-1})^2}{x^2}\frac{\mathrm{d}x}{2\sqrt{x-1}} = 2\int \frac{u^2}{(u^2+1)^2}\,\mathrm{d}u = 2\int \frac{Au+B}{u^2+1}\,\mathrm{d}u + 2\int \frac{Cu+D}{(u^2+1)^2}\,\mathrm{d}u,$$

where $u^2 = (Au+B)(u^2+1) + Cu + D = Au^3 + Bu^2 + (A+C)u + (B+D)$, so $A = C = 0$, $B = 1$ and $D = -1$. Using $u = \tan\theta = \sqrt{x-1}$ gives $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$, $u^2+1 = \sec^2\theta = x$ and

$$\int \frac{\sqrt{x-1}}{x^2}\,\mathrm{d}x = 2\int \frac{\mathrm{d}u}{u^2+1} - 2\int \frac{\mathrm{d}u}{(u^2+1)^2} = 2\int \mathrm{d}\theta - 2\int \cos^2\theta\,\mathrm{d}\theta = \int \mathrm{d}\theta - \int \cos 2\theta\,\mathrm{d}\theta$$
$$= \theta - \sin\theta\cos\theta + D = \tan^{-1}\sqrt{x-1} - \frac{\sqrt{x-1}}{x} + D,$$

since $\sin\theta\cos\theta = \dfrac{\tan\theta}{\sec^2\theta}$. Thus,

$$\int \frac{1+\sqrt{(x-1)^3}}{x^2(x-1)}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{x^2(x-1)} + \int \frac{\sqrt{x-1}}{x^2}\,\mathrm{d}x = \ln\left|\frac{x-1}{x}\right| + \tan^{-1}\sqrt{x-1} + \frac{1-\sqrt{x-1}}{x} + C.$$

2. $\displaystyle\int \csc x\cot^2 x\,\mathrm{d}x$

Since $\cot^2 x + 1 = \csc^2 x$, $\displaystyle\int \csc x\cot^2 x\,\mathrm{d}x = \int \csc^3 x\,\mathrm{d}x - \int \csc x\,\mathrm{d}x$. For the first integral, using $u = \csc x$ gives $\mathrm{d}v = \csc^2 x\,\mathrm{d}x$, $\mathrm{d}u = -\csc x\cot x\,\mathrm{d}x$, $v = -\cot x$ and

$$\int \csc^3 x\,\mathrm{d}x = -\cot x\csc x - \int \csc x\cot^2 x\,\mathrm{d}x = \int \csc x\,\mathrm{d}x - \int \csc^3 x\,\mathrm{d}x - \cot x\csc x,$$

thus $\displaystyle\int \csc^3 x\,\mathrm{d}x = \frac{1}{2}\int \csc x\,\mathrm{d}x - \frac{1}{2}\cot x\csc x$. For the second integral,

$$\int \csc x\,\mathrm{d}x = \int \csc x\frac{\csc x + \cot x}{\csc x + \cot x}\,\mathrm{d}x = -\int \frac{\mathrm{d}(\csc x + \cot x)}{\csc x + \cot x} = -\ln|\csc x + \cot x| + C.$$

Thus, $\displaystyle\int \csc x\cot^2 x\,\mathrm{d}x = -\frac{1}{2}\left[\cot x\csc x + \ln|\csc x + \cot x|\right] + C$.

3. $\displaystyle\int_0^1 x^2\sqrt{x^2+1}\,\mathrm{d}x$

Using $x=\tan\theta$ gives $\mathrm{d}x=\sec^2\theta\,\mathrm{d}\theta$, $\sqrt{x^2+1}=\sec\theta$ and, since $\sec^2 x = 1+\tan^2 x$,

$$\int_0^1 x^2\sqrt{x^2+1}\,\mathrm{d}x = \int_0^{\pi/4}\tan^2\theta\sec^3\theta\,\mathrm{d}\theta = \int_0^{\pi/4}\sec^5\theta\,\mathrm{d}\theta - \int_0^{\pi/4}\sec^3\theta\,\mathrm{d}\theta.$$

For the first integral, the reduction formula gives $\displaystyle\int\sec^5\theta\,\mathrm{d}\theta = \frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{4}\int\sec^3\theta\,\mathrm{d}\theta$. For the second integral, the reduction formula gives $\displaystyle\int\sec^3\theta\,\mathrm{d}\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\,\mathrm{d}\theta$, where $\displaystyle\int\sec\theta\,\mathrm{d}\theta = \ln|\sec\theta+\tan\theta| + C$. Thus,

$$\begin{aligned}\int_0^1 x^2\sqrt{x^2+1}\,\mathrm{d}x &= \left[\frac{1}{4}\sec^3\theta\tan\theta - \frac{1}{8}\sec\theta\tan\theta - \frac{1}{8}\ln|\sec\theta+\tan\theta|\right]_0^{\pi/4}\\ &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8} - \frac{1}{8}\ln|1+\sqrt{2}| = \frac{3\sqrt{8}+\ln|\sqrt{2}-1|}{8}.\end{aligned}$$

4. $\displaystyle\int\frac{\mathrm{d}x}{\sin x(1-\cos x)}$

Using $t=\tan\left(\dfrac{x}{2}\right)$ gives $\cos x = \dfrac{1-t^2}{1+t^2}$, $\sin x = \dfrac{2t}{1+t^2}$, $\mathrm{d}x = \dfrac{2\,\mathrm{d}t}{1+t^2}$ and

$$\begin{aligned}\int\frac{\mathrm{d}x}{\sin x(1-\cos x)} &= \int\frac{2\,\mathrm{d}t/(1+t^2)}{2t(1-[(1-t^2)/(1+t^2)])/(1+t^2)} = \int\frac{\mathrm{d}t}{t(2t^2)/(1-t^2)} = \frac{1}{2}\int\frac{1+t^2}{t^3}\,\mathrm{d}t\\ &= \frac{1}{2}\int\frac{\mathrm{d}t}{t^3} + \frac{1}{2}\int\frac{\mathrm{d}t}{t} = \frac{1}{2}\ln t - \frac{1}{4t^2} + C = \frac{1}{2}\ln\left[\tan\left(\frac{x}{2}\right)\right] - \frac{1}{4}\cot\left(\frac{x}{2}\right) + C.\end{aligned}$$

Approximate the area bound by $y = f(x)$, $x = a$, $x = b$ and the x-axis, given by $\displaystyle\int_a^b f(x)\,\mathrm{d}x$, partitioning the interval $[a, b]$ into n intervals of equal width, $[x_{k-1}, x_k]$, $1 \le k \le n$, with $a = x_0$ and $b = x_n$ and $h = \dfrac{b-a}{n} = x_k - x_{k-1}$.

Trapezoidal Rule

Approximate $f(x)$ with a piecewise-linear function $g(x)$,

$$g(x) = g_k(x) = a_k x + b_k \text{ on } [x_{k-1}, x_k], \quad 1 \le k \le n, \qquad f(x_m) = g(x_m), \quad 0 \le m \le n.$$

The approximation of the definite integral is, thus,

$$\int_a^b f(x)\,\mathrm{d}x \approx \int_a^b g(x)\,\mathrm{d}x = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} g(x)\,\mathrm{d}x = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} g_k(x)\,\mathrm{d}x.$$

It can be determined, using Lagrange basis functions, that

$$g_k(x) = f(x_{k-1})\frac{x - x_k}{x_{k-1} - x_k} + f(x_k)\frac{x - x_{k-1}}{x_k - x_{k-1}} = \frac{f(x_k)(x - x_{k-1}) + f(x_{k-1})(x_k - x)}{h},$$

so
$$\begin{aligned}\int_{x_{k-1}}^{x_k} g_k(x)\,\mathrm{d}x &= \frac{f(x_k)}{h}\int_{x_{k-1}}^{x_k}(x - x_{k-1})\,\mathrm{d}x + \frac{f(x_{k-1})}{h}\int_{x_{k-1}}^{x_k}(x_k - x)\,\mathrm{d}x \\ &= \frac{f(x_k)}{h}\left[\frac{(x - x_{k-1})^2}{2}\right]_{x_{k-1}}^{x_k} - \frac{f(x_{k-1})}{h}\left[\frac{(x_k - x)^2}{2}\right]_{x_{k-1}}^{x_k} = \frac{f(x_k)}{h}\left[\frac{h^2}{2}\right] + \frac{f(x_{k-1})}{h}\left[\frac{h^2}{2}\right] \\ &= \frac{h}{2}[f(x_{k-1}) + f(x_k)],\end{aligned}$$

which is the area of the trapezoid made by $y = g_k(x)$, $x = x_{k-1}$, $x = x_k$ and the x-axis. Thus, the Trapezoidal Rule uses the approximation

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{k=1}^{n}\int_{x_{k-1}}^{x_k} g_k(x)\,\mathrm{d}x = \sum_{k=1}^{n}\frac{h}{2}[f(x_{k-1}) + f(x_k)] = h\left[\frac{f(a) + f(b)}{2} + \sum_{k=1}^{n-1} f(x_k)\right].$$

The error bound for the approximation is

$$\left|\int_a^b f(x)\,\mathrm{d}x - h\left[\frac{f(a) + f(b)}{2} + \sum_{k=1}^{n-1} f(x_k)\right]\right| < \frac{M(b-a)^3}{12n^2} = \frac{Mh^2(b-a)}{12} \text{ where } M = \max_{a \le x \le b}\{|f''(x)|\}.$$

Simpson's 1 / 3 Rule

Let n be even. Approximate $f(x)$ with a piecewise-parabolic function $g(x)$,

$$g(x) = g_k(x) = a_k x^2 + b_k x + c_k \text{ on } [x_{2k-2}, x_{2k}], \quad 1 \le k \le n/2, \qquad f(x_m) = g(x_m), \quad 0 \le m \le n.$$

The approximation of the definite integral is, thus,

$$\int_a^b f(x)\,\mathrm{d}x \approx \int_a^b g(x)\,\mathrm{d}x = \sum_{k=1}^{n/2}\int_{x_{2k-2}}^{x_{2k}} g(x)\,\mathrm{d}x = \sum_{k=1}^{n/2}\int_{x_{2k-2}}^{x_k} g_k(x)\,\mathrm{d}x.$$

It can be determined, using Lagrange basis functions, that

$$\begin{aligned} g_k(x) &= f(x_{2k-2})\frac{(x-x_{2k})(x-x_{2k-1})}{(x_{2k-2}-x_{2k})(x_{2k-2}-x_{2k-1})} + f(x_{2k-1})\frac{(x-x_{2k})(x-x_{2k-2})}{(x_{2k-1}-x_{2k})(x_{2k-1}-x_{2k-2})} \\ &\quad + f(x_{2k})\frac{(x-x_{2k-1})(x-x_{2k-2})}{(x_{2k}-x_{2k-1})(x_{2k}-x_{2k-2})} \end{aligned}$$

$$\begin{aligned} g_k(x) &= \frac{f(x_{2k-2})}{2h^2}(x_{2k-1}+h-x)(x_{2k-1}-x) + \frac{f(x_{2k-1})}{h^2}(x_{2k-1}+h-x)(x-x_{2k-1}+h) \\ &\quad + \frac{f(x_{2k})}{2h^2}(x-x_{2k-1})(x-x_{2k-1}+h), \end{aligned}$$

so

$$\begin{aligned} \int_{x_{2k-2}}^{x_{2k}} g_k(x)\,\mathrm{d}x &= \frac{f(x_{2k-2})}{2h^2}\int_{x_{2k-1}-h}^{x_{2k-1}+h}[x^2-(2x_{2k-1}+h)x + x_{2k-1}(x_{2k-1}+h)]\,\mathrm{d}x \\ &\quad - \frac{f(x_{2k-1})}{h^2}\int_{x_{2k-1}-h}^{x_{2k-1}+h}[x^2-2x_{2k-1}x+x_{2k-1}^2-h^2]\,\mathrm{d}x \\ &\quad + \frac{f(x_{2k})}{2h^2}\int_{x_{2k-1}-h}^{x_{2k-1}+h}[x^2-(2x_{2k-1}-h)x + x_{2k-1}(x_{2k-1}-h)]\,\mathrm{d}x \\ &= \frac{f(x_{2k-2})}{2h^2}\left[\frac{x^3}{3}-\frac{(2x_{2k-1}+h)x^2}{2}+x_{2k-1}(x_{2k-1}+h)x\right]_{x_{2k-1}-h}^{x_{2k-1}+h} \\ &\quad - \frac{f(x_{2k-1})}{h^2}\left[\frac{x^3}{3}-x_{2k-1}x^2+(x_{2k-1}^2-h^2)x\right]_{x_{2k-1}-h}^{x_{2k-1}+h} \\ &\quad + \frac{f(x_{2k})}{2h^2}\left[\frac{x^3}{3}-\frac{(2x_{2k-1}-h)x^2}{2}+x_{2k-1}(x_{2k-1}-h)x\right]_{x_{2k-1}-h}^{x_{2k-1}+h} \\ &= \frac{f(x_{2k-2})-2f(x_{2k-1})+f(x_{2k})}{6h^2}[(x_{2k-1}+h)^3-(x_{2k-1}-h)^3] \\ &\quad - \frac{f(x_{2k-2})(2x_{2k-1}+h)-4f(x_{2k-1})x_{2k-1}+f(x_{2k})(2x_{2k-1}-h)}{4h^2}[(x_{2k-1}+h)^2-(x_{2k-1}-h)^2] \\ &\quad + \frac{f(x_{2k-2})x_{2k-1}(x_{2k-1}+h)-2f(x_{2k-1})(x_{2k-1}^2-h^2)+f(x_{2k})x_{2k-1}(x_{2k-1}-h)}{h} \\ &= \frac{f(x_{2k-2})-2f(x_{2k-1})+f(x_{2k})}{3h}(3x_{2k-1}^2+h^2) \\ &\quad - \frac{f(x_{2k-2})(2x_{2k-1}+h)-4f(x_{2k-1})x_{2k-1}+f(x_{2k})(2x_{2k-1}-h)}{h}x_{2k-1} \\ &\quad + \frac{f(x_{2k-2})x_{2k-1}(x_{2k-1}+h)-2f(x_{2k-1})(x_{2k-1}^2-h^2)+f(x_{2k})x_{2k-1}(x_{2k-1}-h)}{h} \\ &= \frac{f(x_{2k-2})}{3h}[(3x_{2k-1}^2+h^2)-3(2x_{2k-1}^2+x_{2k-1}h)+3x_{2k-1}(x_{2k-1}+h)] \\ &\quad + \frac{f(x_{2k-1})}{3h}[-2(3x_{2k-1}^2+h^2)+12x_{2k-1}^2-6(x_{2k-1}^2-h^2)] \\ &\quad + \frac{f(x_{2k})}{3h}[(3x_{2k-1}^2+h^2)-3(2x_{2k-1}^2-x_{2k-1}h)+3x_{2k-1}(x_{2k-1}-h)] \\ &= \frac{h}{3}f(x_{2k-2}) + \frac{4h}{3}f(x_{2k-1}) + \frac{h}{3}f(x_{2k}) = \frac{h}{3}[f(x_{2k-2})+4f(x_{2k-1})+f(x_{2k})]. \end{aligned}$$

Thus, Simpson's 1 / 3 Rule uses the approximation

$$\begin{aligned} \int_a^b f(x)\,\mathrm{d}x &\approx \sum_{k=1}^{n/2}\int_{x_{2k-2}}^{x_{2k}} g_k(x)\,\mathrm{d}x = \sum_{k=1}^{n/2}\frac{h}{3}[f(x_{2k-2})+4f(x_{2k-1})+f(x_{2k})] \\ &\approx \frac{h}{3}\left[f(a)+f(b)+\sum_{k=1}^{n-1}[3-(-1)^k]f(x_k)\right]. \end{aligned}$$

The error bound for the approximation is given by

$$\left| \int_a^b f(x)\,\mathrm{d}x - \frac{h}{3}\left[f(a) + f(b) + \sum_{k=1}^{n-1} [3 - (-1)^k] f(x_k) \right] \right| < \frac{M(b-a)^5}{180n^4} = \frac{Mh^4(b-a)}{180}$$

where $M = \max\limits_{a \le x \le b} \{|f^{(4)}(x)|\}$.

Name: ______________________________

§7.5 #58 Evaluate the integral $\int \frac{x \ln x}{\sqrt{x^2-1}}\,\mathrm{d}x$.

Letting $u = \ln x$ makes $\mathrm{d}v = \frac{x\,\mathrm{d}x}{\sqrt{x^2-1}}$, giving $v = \sqrt{x^2-1}$ and $\mathrm{d}u = \frac{\mathrm{d}x}{x}$: letting $x = \sec\theta$ gives $\mathrm{d}x = \sec\theta\tan\theta\,\mathrm{d}\theta$, $\sqrt{x^2-1} = \tan\theta$ and

$$\int \frac{x\ln x}{\sqrt{x^2-1}}\,\mathrm{d}x = (\ln x)\sqrt{x^2-1} - \int \frac{\sqrt{x^2-1}}{x}\,\mathrm{d}x = (\ln x)\sqrt{x^2-1} - \int \frac{\sec\theta\tan^2\theta\,\mathrm{d}\theta}{\sec\theta}$$
$$= (\ln x)\sqrt{x^2-1} - \int \sec^2\theta\,\mathrm{d}\theta + \int \mathrm{d}\theta = (\ln x - 1)\sqrt{x^2-1} + \cos^{-1}\frac{1}{x} + C.$$

§11.1 #90 Let $a_n = \left(1+\frac{1}{n}\right)^n$.

(a) Show that if $0 \le a < b$, then $\frac{b^{n+1}-a^{n+1}}{b-a} < (n+1)b^n$.
As can be confirmed by synthetic division,

$$b^{n+1}-a^{n+1} = (b-a)(b^n+ab^{n-1}+\cdots+a^{n-1}b+a^n) < (b-a)(b^n+b^n+\cdots+b^n+b^n) = (n+1)(b-a)b^n.$$

Dividing both sides by the positive value $b-a$ gives the inequality.

(b) Deduce that $b^n[(n+1)a-nb] < a^{n+1}$.
From above,

$$b^{n+1}-a^{n+1} < (n+1)(b^{n+1}-ab^n) \implies a^{n+1} > (n+1)ab^n+b^{n+1}-(n+1)b^{n+1} = b^n[(n+1)a-nb].$$

(c) Use $a = 1+\frac{1}{n+1}$ and $b = 1+\frac{1}{n}$ in part (b) to show that $\{a_n\}$ is increasing.
Noting, for $n \in \mathbb{N}$, $0 < a < b$,

$$a_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n\left[(n+1)\left(1+\frac{1}{n+1}\right) - n\left(1+\frac{1}{n}\right)\right] = a_n[n+1-n] = a_n.$$

Thus, $\{a_n\}$ is increasing.

(d) Use $a = 1$ and $b = 1+\frac{1}{2n}$ in part (b) to show that $a_{2n} < 4$.
Noting, for $n \in \mathbb{N}$, $0 < a < b$,

$$1^{n+1} > \left(1+\frac{1}{2n}\right)^n\left[(n+1)1 - n\left(1+\frac{1}{2n}\right)\right] = \sqrt{a_{2n}}\left[n+1-n-\frac{1}{2}\right] = \sqrt{a_{2n}}\left[n+1-n-\frac{1}{2}\right].$$

Thus, $\sqrt{a_{2n}} < 2$ and $a_{2n} < 4$.

(e) Use parts (c) and (d) to show that $a_n < 4$ for all n.
For every odd term of the sequence, $a_{2n-1} < a_{2n} < 4$.

(f) Use Theorem 12 to show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists. (The limit is e.)
Since $\{a_n\}$ is increasing, and bounded above by 4, by the Bounded Monotone Sequence Theorem, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists.

§11.1 #92 (a) Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.
Using the ε–δ definition, the given limits indicate that for every $\varepsilon > 0$, there exists $N_0, N_1 \in \mathbb{N}$, such that, for all $n_i > N_i$, $|a_{2n_i-i} - L| < \varepsilon$, $i = 0, 1$. Thus, for the given ε, for every $n > N_\varepsilon = 2\max\{N_0, N_1\}$, $|a_n - L| < \varepsilon$—by definition, $\lim_{n\to\infty} a_n = L$, and $\{a_n\}$ is convergent.

(b) If $a_1 = 1$ and $a_{n+1} = 1 + \dfrac{1}{1+a_n}$, find the first eight terms of the sequence $\{a_n\}$. Then use part (a) to show that $\lim_{n\to\infty} a_n = \sqrt{2}$. This gives the *continued fraction expansion* $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2+\cdots}}$.

The first eight terms of the sequence are $1, \frac{3}{2} = 1.5, \frac{7}{5} = 1.4, \frac{17}{12} = 1.41\bar{6}, \frac{41}{29} \approx 1.4137931, \frac{99}{70} = 1.4\overline{142857}, \frac{239}{169} \approx 1.4142, \frac{577}{408} \approx 1.4142156862745.$

Using the recursion, $a_{n+2} = \dfrac{3a_n+4}{2a_n+3} = \dfrac{3}{2} - \dfrac{1}{4a_n+6}$, with $a_1 = 1$ and $a_2 = \dfrac{3}{2}$. Consider $a_{n+2} - a_n = \dfrac{4-2a_n^2}{2a_n+3}$—this indicates that $a_{n+2} > a_n$ when $|a_n| < \sqrt{2} \approx 1.414213562373$ and $a_{n+2} < a_n$ when $|a_n| > \sqrt{2}$. Also, $a_{n+2} - \sqrt{2} = \dfrac{2(3-2\sqrt{2})(a_n - \sqrt{2})}{4(a_n - \sqrt{2}) + 4\sqrt{2} + 6}$—this indicates that, if $a_n > \sqrt{2}$, then $a_{n+2} > \sqrt{2}$, or if $|a_n| < \sqrt{2}$, then $|a_{n+2}| < \sqrt{2}$.

Thus, $\{a_{2n-1}\}$ is an increasing sequence bounded above by $\sqrt{2}$ and $\{a_{2n}\}$ is a decreasing sequence bounded below by $\sqrt{2}$—both subsequences are thus convergent. Assuming that $a_n \to L$, $L = \dfrac{3L+4}{2L+3}$, then $2L^2 = 4$ or $L = \pm\sqrt{2}$—since the recurrence works for both subsequences, $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = \sqrt{2}$, and, from above, $\lim_{n\to\infty} a_n = \sqrt{2}$.

§11.2 #54 Express $7.\overline{12345}$ as a ratio of integers.

By definition, $7.\overline{12345} = 7 + \dfrac{12345}{100000} \sum_{n=0}^{\infty} \dfrac{1}{100000} = 7 + \dfrac{12345}{99999} = \dfrac{237446}{33333}$.

§11.2 #64 We have seen that the harmonic series is a divergent series whose terms approach 0. Show that $\sum_{n=1}^{\infty} \ln\left(1 + \dfrac{1}{n}\right)$ is another series with this property.

The partial sum $s_N = \sum_{n=1}^{N} \ln\left(\dfrac{n+1}{n}\right) = \ln\left[\prod_{n=1}^{N} \dfrac{n+1}{n}\right] = \ln(N+1)$ is unbounded as $N \to \infty$, but $a_n = \ln\left(\dfrac{n+1}{n}\right)$ approaches $\ln 1 = 0$ as $n \to \infty$.

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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§7.5 #77 Evaluate $\displaystyle\int \frac{xe^x}{\sqrt{1+e^x}}\,\mathrm{d}x$.

Using $u = e^x$ gives $\mathrm{d}u = e^x\,\mathrm{d}x$, $x = \ln u$ and $\displaystyle\int \frac{xe^x}{\sqrt{1+e^x}}\,\mathrm{d}x = \int \frac{u\ln u}{\sqrt{1+u}}\,\mathrm{d}u$. Using $v = \sqrt{u+1}$ gives $\mathrm{d}v = \dfrac{\mathrm{d}u}{2\sqrt{u+1}}$, $u = v^2 - 1$ and

$$\int \frac{u\ln u}{\sqrt{1+u}}\,\mathrm{d}u = 2\int (v^2-1)\ln(v^2-1)\,\mathrm{d}v = 2\int (v^2-1)\ln(v-1)\,\mathrm{d}v + 2\int (v^2-1)\ln(v+1)\,\mathrm{d}v.$$

If $w_\pm = \ln(v \pm 1)$, then $\mathrm{d}z = (v^2-1)\,\mathrm{d}v$, giving $\mathrm{d}w_\pm = \dfrac{\mathrm{d}v}{v\pm 1}$, $z = \dfrac{v^3-3v}{3}$ and

$$\begin{aligned}
\int \frac{u\ln u}{\sqrt{1+u}}\,\mathrm{d}u &= \frac{2}{3}(v^3-3v)\ln(v^2-1) - \frac{2}{3}\int \frac{(v^3-3v)\,\mathrm{d}v}{v+1} - \frac{2}{3}\int \frac{(v^3-3v)\,\mathrm{d}v}{v-1} \\
&= \frac{2}{3}(v^3-3v)\ln(v^2-1) - \frac{4}{3}\int (v^2-2)\,\mathrm{d}v - \frac{4}{3}\int \frac{\mathrm{d}v}{v+1} + \frac{4}{3}\int \frac{\mathrm{d}v}{v-1} \\
&= \frac{2}{3}(v^3-3v)\ln(v^2-1) - \frac{4}{9}(v^3-6v) + \frac{4}{3}\ln\left|\frac{v-1}{v+1}\right| + C \\
&= \frac{2}{3}x[(e^x-2)\sqrt{1+e^x}+2] - \frac{4}{9}(e^x-5)\sqrt{1+e^x} - \frac{8}{3}\ln(\sqrt{1+e^x}+1) + C.
\end{aligned}$$

§11.1 #83 (a) Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age two months. If we start with one newborn pair, how many pairs of rabbits will we have in the nth month? Show that the answer is f_n, where

> *The Fibonacci sequence* $\{f_n\}$ is defined recursively by the conditions $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$. Each term is the sum of the two preceding terms.

In the first month, there is one nonproductive pair of age less than one month, so $f_1 = 1$. In the second month, there is one nonproductive pair of age less than two months, so $f_2 = 1$. In the third month, the productive pair produces a nonproductive pair of age less than one month, so $f_3 = 2 = f_1 + f_2$—for $n = 3$, f_{n-2} counts the productive pairs (which adds f_{n-2} to the total for the nth month) and $f_{n-1} - f_{n-2}$ counts the nonproductive pairs, so in the nth month, there are f_{n-2} productive pairs of age two months or more, $f_{n-1} - f_{n-2}$ nonproductive pairs of age between one and two months, and f_{n-2} new pairs of age less than one month, giving $f_n = f_{n-2} + f_{n-1} - f_{n-2} + f_{n-2} = f_{n-1} + f_{n-2}$. For example, in the fourth month, there is $f_2 = 1$ productive pair of age more than two months (the initial pair, now aged between three and four months), $f_3 - f_2 = f_1 = 1$ nonproductive pair of age between one and two months, and $f_2 = 1$ new pair of age less than one month, so $f_4 = 1 + 1 + 1 = 3$.

(b) Let $a_n = \dfrac{f_{n+1}}{f_n}$ and show that $a_{n-1} = 1 + \dfrac{1}{a_{n-2}}$. Assuming $\{a_n\}$ is convergent, find its limit.

$a_1 = 1$ and $a_2 = 2$: for $n > 2$, $a_{n-1} = \dfrac{f_n}{f_{n-1}} = \dfrac{f_{n-1}+f_{n-2}}{f_{n-1}} = 1 + \dfrac{1}{f_{n-1}/f_{n-2}} = 1 + \dfrac{1}{a_{n-2}}$. If $\displaystyle\lim_{n\to\infty} a_n = L$, then $\displaystyle\lim_{n\to\infty} a_{n-1} = L = 1 + \left[\lim_{n\to\infty} a_{n-2}\right]^{-1} = 1 + \frac{1}{L}$ so $L^2 = L + 1$ or $L^2 - L - 1 = 0$, so $L = \dfrac{1\pm\sqrt{5}}{2}$. Since $a_1 > 0$, $a_2 > 0$—it follows that if $a_n > 0$ then $a_{n+1} > 0$, so $\displaystyle\lim_{n\to\infty} a_n > 0$, and $L = \dfrac{1+\sqrt{5}}{2} = \phi$, known as the golden ratio. (The other root is denoted as $\hat{\phi} = \dfrac{1-\sqrt{5}}{2}$ and it can be shown that $f_n = \dfrac{1}{\sqrt{5}}(\phi^n - \hat{\phi}^n)$.)

§11.1 #91 Let a and b be positive integers with $a > b$. Let a_1 be their arithmetic mean and b_1 their geometric mean: $a_1 = \dfrac{a+b}{2}$, $b_1 = \sqrt{ab}$. Repeat this process so that, in general, $a_{n+1} = \dfrac{a_n + b_n}{2}$, $b_{n+1} = \sqrt{a_n b_n}$.

(a) Use mathematical induction to show that $a_n > a_{n+1} > b_{n+1} > b_n$.

It suffices to show that if $x > y > 0$, $x > \frac{x+y}{2} > \sqrt{xy} > y$. Since $x > y$, $\frac{x}{2} > \frac{y}{2}$, so $x = \frac{x+x}{2} > \frac{x+y}{2} = \frac{y+x}{2} > \frac{y+y}{2} = y$. Since $x > y > 0$, $\sqrt{x} > \sqrt{y}$, so $x = \sqrt{x \cdot x} > \sqrt{x \cdot y} = \sqrt{y \cdot x} > \sqrt{y \cdot y} = y$. Finally, $0 < \left(\frac{\sqrt{x} - \sqrt{y}}{2}\right)^2 = \frac{x+y}{2} - \sqrt{xy}$, so $\frac{x+y}{2} > \sqrt{xy}$.

(b) Deduce that both $\{a_n\}$ and $\{b_n\}$ are convergent.

$\{a_n\}$ is decreasing, and $\{b_n\}$ is increasing and both are bounded above by a) and bounded below by b. By the Monotone Bounded Sequence Theorem, both are convergent sequences.

(c) Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Gauss called the common value of these limits the *arithmetic-geometric mean* of the numbers a and b.

$\{c_n = a_n - b_n\}$ is a decreasing sequence, bounded above by $a - b$ and below by 0; by the Monotone Bounded Sequence Theorem, it is convergent. Since $c_{n+1} = a_{n+1} - b_{n+1} = \frac{a_n + b_n}{2} - \sqrt{a_n b_n} < \frac{a_n - b_n}{2} = \frac{c_n}{2}$, thus $c_n < \frac{a-b}{2^n}$, and by the squeeze theorem, $0 \le \lim_{n\to\infty} c_n \le \lim_{n\to\infty} \frac{a-b}{2^n} = 0$, so $\lim_{n\to\infty}(a_n - b_n) = 0$ and the conclusion follows from that fact that both limits exist.

§11.2 #77 In Example 9 we showed that the harmonic series is divergent. Here we outline another method, making use of the fact that $e^x > 1 + x$ for any $x > 0$. (See §4.3 #84.)

If $f(x) = e^x - x - 1$, $f(0) = 0$ and $f'(x) = e^x - 1 > 0$ when $x > 0$, so $f(x) > 0$ and $e^x > 1 + x$ for $x > 0$.

If s_n is the nth partial sum of the harmonic series, show that $e^{s_n} > n + 1$. Why does this imply that the harmonic series is divergent?

Since $e^{s_n} = \prod_{k=1}^{n} e^{1/k} > \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) = \frac{2}{1} \cdot \frac{3}{2} \cdot \dots \cdot \frac{n+1}{n} = n + 1$, as $s_n > \ln(n+1)$, if the harmonic series was convergent to s, since $\{s_n\}$ is increasing, $s > s_n$, and the sequence $\{\ln(n+1)\}$, which is increasing, would be bounded above by s and below by 0, and would be convergent—but $\{\ln(n+1)\}$ is divergent, so $\{s_n\}$ cannot be convergent, and the harmonic series diverges.

§11.2 #91 Consider the series $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

(a) Find the partial sums s_1, s_2, s_3, and s_4. Do you recognize the denominators? Use the pattern to guess a formula for s_n.

$s_1 = \frac{1}{2}$, $s_2 = s_1 + \frac{1}{3} = \frac{5}{6}$, $s_3 = s_2 + \frac{1}{8} = \frac{23}{24}$ and $s_4 = s_3 + \frac{1}{30} = \frac{119}{120}$—that is, for $n = 1, 2, 3, 4$, $s_n = 1 - \frac{1}{(n+1)!}$.

(b) Use mathematical induction to prove your guess.

Assume that $s_n = 1 - \frac{1}{(n+1)!}$ for $n = 1, \dots, k$. Then $s_{k+1} = s_k + \frac{k+1}{(k+2)!} = 1 - \frac{k+2}{(k+2)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$. This shows that if the formula holds for all $n = 1, \dots, k$, it holds for $n = k + 1$—by mathematical induction, this shows that the formula holds for $n \in \mathbb{N}$.

(c) Show that the given infinite series is convergent, and find its sum.

$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \lim_{n\to\infty} s_n = 1 - \lim_{n\to\infty} \frac{1}{(n+1)!} = 1.$

Given a sequence $\{a_n\}_{n=0}^{\infty}$, determine properties of the series $\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} s_N$, where the *partial sum* s_N is defined as $s_N = \sum_{n=0}^{N} a_n$. ($\star$ indicates use in power series, **Estimate**s after **Test**s for convergence)

§11.1 $\{s_N\}_{N=0}^{\infty}$ is a sequence: if it converges, $s = \lim_{N\to\infty} s_N$, converges to the series; otherwise, the series diverges

- $s_N = f(N)$: $\lim_{N\to\infty} s_N = \lim_{x\to\infty} f(x)$—both limits exist (converge) or do not exist (diverge)
- $\lim_{N\to\infty} |s_N| = 0$: $\lim_{N\to\infty} s_N = 0$
- $|r| < 1$: $r^n \to 0$; $r = 1$: $r^n \to 1$
- every monotone (increasing $s_{N+1} \geq s_N$ or decreasing $s_{N+1} \leq s_N$) bounded (there exist L, R such that $L \leq s_N \leq R$) sequence converges

§11.2–3 the convergence of some series can be determined by the sequence $\{a_n\}_{n=0}^{\infty}$

Test/Formula $a_n = ar^n$: $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$ if $|r| < 1$, otherwise the series diverges (geometric series)

Test $a_n = \dfrac{1}{n^p}$: $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges if $p > 1$, otherwise the series diverges (p-series)

Test $\lim_{n\to\infty} a_n \neq 0$: $\sum_{n=0}^{\infty} a_n$ diverges (divergence test)

§11.3–4 $\sum_{n=0}^{\infty} a_n$ can be compared with something known to converge or diverge

Test $a_n = f(n)$, $f(x) > 0$, descending, continuous on $x \geq 1$: $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_1^t f(x)\,\mathrm{d}x$ both converge or diverge (integral test)

Estimate $\sum_{n=1}^{N} f(n) + \int_{N+1}^{\infty} f(x)\,\mathrm{d}x \leq \sum_{n=1}^{\infty} f(n) \leq \sum_{n=1}^{N} f(n) + \int_{N}^{\infty} f(x)\,\mathrm{d}x$ (integral error estimate)

Test $0 \leq a_n \leq b_n$ and $\sum_{n=0}^{\infty} b_n$ converges: $\sum_{n=0}^{\infty} a_n$ converges (comparison test)

Test $0 \leq b_n \leq a_n$ and $\sum_{n=0}^{\infty} b_n$ diverges: $\sum_{n=0}^{\infty} a_n$ diverges (comparison test)

Test $0 \leq a_n, b_n$ and $\lim_{n\to\infty} \dfrac{b_n}{a_n} = c > 0$: $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge or diverge (limit comparison test)

$\star$ §11.5 if $a_n = (-1)^n b_n$, $b_n > 0$ (*alternating series*)

Test $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$: $\sum_{n=0}^{\infty} (-1)^n b_n$ converges (alternating series test)

Estimate $\left|\sum_{n=N+1}^{\infty} (-1)^n b_n\right| = \left|\sum_{n=0}^{\infty} (-1)^n b_n - \sum_{n=0}^{N} (-1)^n b_n\right| < b_{N+1}$ (alternating series error estimate)

§11.6 other trends of a_n as $n \to \infty$ may determine convergence of $\sum_{n=0}^{\infty} a_n$

$\star$ **Test** $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| < 1$: $\sum_{n=0}^{\infty} a_n$ converges; $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| > 1$: $\sum_{n=0}^{\infty} a_n$ diverges (ratio test)

Test $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$: $\sum_{n=0}^{\infty} a_n$ converges; $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$: $\sum_{n=0}^{\infty} a_n$ diverges (root test)

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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§7.5 #46 Evaluate $\int \frac{(x-1)e^x}{x^2}\,\mathrm{d}x$.

Using $u = e^x$ and $\mathrm{d}v = \frac{\mathrm{d}x}{x^2}$ gives $v = -\frac{1}{x}$, $\mathrm{d}u = e^x\,\mathrm{d}x$ and $\int \frac{(x-1)e^x}{x^2}\,\mathrm{d}x = \int \frac{e^x}{x}\,\mathrm{d}x - \int \frac{e^x}{x^2}\,\mathrm{d}x = \frac{e^x}{x} + C$.

§11.3 #38 Find the sum of the series $\sum_{n=1}^{\infty} ne^{-2n}$ correct to four decimal places.

The sum is $\sum_{n=1}^{\infty} f(n)$, where $f(x) = \frac{x}{e^{2x}} > 0$ and $f'(x) = \frac{1-2x}{e^{2x}} < 0$, where $x > 0$. By integration by parts, $\int \frac{x\,\mathrm{d}x}{e^{2x}} = -\frac{x}{2e^{2x}} + \frac{1}{2}\int \frac{\mathrm{d}x}{e^{2x}} = -\frac{2x+1}{4e^{2x}} + C$. By the integral test, applying L'Hôpital's Rule, $\int_1^{\infty} \frac{x\,\mathrm{d}x}{e^{2x}} = \lim_{t\to\infty}\left[-\frac{2x+1}{4e^{2x}}\right]_1^t = \frac{3}{4e^2} - \lim_{t\to\infty}\frac{2t+1}{4e^{2t}} = \frac{3}{4e^2} - \lim_{t\to\infty}\frac{2}{8e^{2t}} = \frac{3}{4e^2}$, indicating that the series is convergent. To generate a sufficiently accurate approximation of the sum, $n \in \mathbb{N}$ must be determined such that

$$\int_n^{n+1} \frac{x\,\mathrm{d}x}{e^{2x}} = \frac{2n+1}{4e^{2n}} - \frac{2n+3}{4e^{2n+2}} = \frac{2ne^2 + e^2 - 2n - 3}{4e^{2n+2}} < \frac{1}{10000} \implies 5000[2n(1-e^{-2}) + 1 - 3e^{-2}] < e^{2n},$$

which gives $4\ln 5 + 2\ln 2 + \ln[2n(1-e^{-2}) + 1 - 3e^{-2}] < 2n$, which is true for $n \geq 6$: the sum can be found in the interval $\sum_{n=1}^{6} ne^{-2n} + \int_7^{\infty} \frac{\mathrm{d}x}{e^{2x}} \leq \sum_{n=1}^{\infty} ne^{-2n} \leq \sum_{n=1}^{6} ne^{-2n} + \int_6^{\infty} \frac{\mathrm{d}x}{e^{2x}}$—that is,

$$\sum_{n=1}^{6} ne^{-2n} = \frac{1}{e^2} + \frac{2}{e^4} + \frac{3}{e^6} + \frac{4}{e^8} + \frac{5}{e^{10}} + \frac{6}{e^{12}} = \frac{e^{10} + 2e^8 + 3e^6 + 4e^4 + 5e^2 + 6}{e^{12}},$$

$$\int_7^{\infty} \frac{\mathrm{d}x}{e^{2x}} = \lim_{t\to\infty}\left[-\frac{2x+1}{4e^{2x}}\right]_7^t = \frac{15}{4e^{14}}, \qquad \int_6^{\infty} \frac{\mathrm{d}x}{e^{2x}} = \lim_{t\to\infty}\left[-\frac{2x+1}{4e^{2x}}\right]_6^t = \frac{13}{4e^{12}},$$

which gives the midpoint of the interval $\frac{8e^{12} + 16e^{10} + 24e^8 + 32e^6 + 40e^4 + 61e^2 + 15}{8e^{14}} \approx 0.18102$ as an approximation to the sum with the desired accuracy. The sum can be determined as such:

$$\sum_{n=1}^{\infty} \frac{n}{e^{2n}} = \sum_{n=1}^{\infty} \frac{1}{e^{2n}} + \sum_{n=2}^{\infty} \frac{1}{e^{2n}} + \sum_{n=3}^{\infty} \frac{1}{e^{2n}} + \cdots = \left(1 + \frac{1}{e^2} + \frac{1}{e^4} + \cdots\right)\sum_{n=1}^{\infty} \frac{1}{e^{2n}} = \left(\sum_{n=0}^{\infty} \frac{1}{e^{2n}}\right)\left(\sum_{n=1}^{\infty} \frac{1}{e^{2n}}\right)$$
$$= \frac{1}{1-e^{-2}}\frac{e^{-2}}{1-e^{-2}} = \frac{e^2}{(e^2-1)^2} \approx 0.1810154.$$

§11.4 #34 Use the sum of the first 10 terms to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^4}$. Estimate the error.

Since $e > 1$, $\{\sqrt[n]{e}\}$ is a decreasing sequence. Every $a_n = \frac{e^{1/n}}{n^4} \leq \frac{e}{n^4} = b_n$, and $\sum_{n=1}^{\infty} b_n = e\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a convergent p-series ($p = 4 > 1$), $s = \sum_{n=1}^{\infty} a_n$ exists. If $s = s_n + r_n$, where $s_n = \sum_{k=1}^{n} \frac{e^{1/k}}{k^4}$,

$$r_n = \sum_{k=n+1}^{\infty} \frac{e^{1/k}}{k^4} \leq \sum_{k=n+1}^{\infty} \frac{e^{1/(n+1)}}{k^4} \leq \sqrt[n+1]{e}\int_n^{\infty} \frac{\mathrm{d}x}{x^4} = \sqrt[n+1]{e}\lim_{t\to\infty}\left[-\frac{1}{3x^3}\right]_n^t = \frac{\sqrt[n+1]{e}}{3n^3}.$$

Thus, $s_{10} \approx 2.847476$ approximates s within $r_{10} < 0.00037$, or correct up to the third decimal place. By `wolframalpha.com`, $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^4} \approx 2.84778$.

§11.5 #36 Use the following steps to show that $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$. Let h_n and s_n be the partial sums of the harmonic and alternating harmonic series.

(a) Show that $s_{2n} = h_{2n} - h_n$.
Given $h_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ and $s_n = \sum\limits_{k=1}^{n} \frac{(-1)^{k-1}}{k}$,

$$s_{2n} = \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} = \sum_{k=1}^{2n} \frac{1-1-(-1)^k}{k} = \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{2n} \frac{1+(-1)^k}{k} = h_{2n} - \sum_{m=1}^{n} \frac{2}{2m} = h_{2n} - h_n.$$

(b) From Exercise 11.3.44, we have $h_n - \ln n \to \gamma$ as $n \to \infty$, and therefore $h_{2n} - \ln(2n) \to \gamma$ as $n \to \infty$. Use these facts together with part (a) to show that $s_{2n} \to \ln 2$ as $n \to \infty$.
From above,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} (h_{2n} - h_n) = \lim_{n\to\infty} [(h_{2n} - \ln n - \ln 2) - (h_n - \ln n) + \ln 2]$$
$$= \lim_{n\to\infty} [h_{2n} - \ln(2n)] - \lim_{n\to\infty} (h_n - \ln n) + \lim_{n\to\infty} \ln 2 = \gamma - \gamma + \ln 2.$$

§11.6 #44 For which positive integers k is the following series convergent? $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$

By the Ratio Test, the series converges when

$$\lim_{n\to\infty} \left| \frac{[(n+1)!]^2/(kn+k)!}{(n!)^2/(kn)!} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{(kn+1)\cdots(kn+k)} < 1.$$

If $k=1$, the limit is $\lim\limits_{n\to\infty} (n+1)$, which is unbounded. (Note that $\frac{(n!)^2}{n!} = n! \to \infty$ when $n \to \infty$.) If $k=2$, the limit is $\lim\limits_{n\to\infty} \frac{n+1}{2(2n+1)} = \frac{1}{4} < 1$. If $k > 2$, the limit is 0, as the denominator has degree k. Thus, the series converges for every positive integer $k \neq 1$.

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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This is an open-book quiz. Work quietly and individually. You may use a calculator, but not any internet-enabled devices. Make sure that there is a seat between you and your neighbors.

For each of the series below, determine if the series converges or not. (Each divergent series is worth 1 point.) If the series converges, determine the sum or an approximation of the sum of the series correct to two decimal places—you do not need to evaluate the approximation. (Each convergent series is worth 2 points.)

1. $3 + \frac{3}{8} + \frac{1}{9} + \frac{3}{64} + \frac{3}{125} + \frac{1}{72} + \cdots$

   This is the series $s = \sum_{n=1}^{\infty} \frac{3}{n^3}$, which is a multiple of convergent p-series ($p = 3$). Since $\int \frac{3\,\mathrm{d}x}{x^3} = \frac{3}{2x^2} + C$, $\int_N^{N+1} \frac{3\,\mathrm{d}x}{x^3} = \frac{3(2N+1)}{2N^2(N+1)^2} < \frac{1}{100}$ gives $150 < N^4 + 2N^3 + N^2 - 300N$, which is true whenever $N \geq 7$, so an approximation is $\sum_{n=1}^{7} \frac{3}{n^3} + \int_8^{\infty} \frac{3\,\mathrm{d}x}{x^3} + \frac{3 \cdot 15}{4 \cdot 7^2 \cdot 8^2} \approx 3.60664622813411$ approximates $3\zeta(3) \approx 3.6061707$.

2. $\frac{1}{3} + \frac{2}{9} + \frac{3}{19} + \frac{4}{33} + \frac{5}{51} + \frac{6}{73} + \cdots$

   This is the series $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$. Since $\int \frac{x\,\mathrm{d}x}{2x^2+1} = \frac{1}{4}\ln(2x^2+1) + C$, thus $\int_1^{\infty} \frac{x\,\mathrm{d}x}{2x^2+1}$ is divergent, as is the series.

3. $\frac{1}{2} - \frac{3}{2} + \frac{9}{4} - \frac{27}{12} + \frac{81}{48} - \frac{243}{240} + \cdots$

   This series is $\frac{1}{2}\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$, which is a convergent alternating series, since by the Ratio Test $\lim_{n\to\infty} \left| \frac{(-3)^{n+1}/2(n+1)!}{(-3)^n/2n!} \right| = \lim_{n\to\infty} \left| \frac{3}{n+1} \right| = 0$. Since $\frac{3^{N+1}}{2(N+1)!} < \frac{1}{200}$ gives $300 \cdot 3^N < (N+1)!$, which is true whenever $N \geq 10$, so an approximation is $\frac{1}{2}\sum_{n=0}^{10} \frac{(-3)^n}{n!} \approx 0.026663 \approx \frac{1}{2e^3} \approx 0.02489$.

4. $\frac{128}{27} + \frac{8}{3} + \frac{3}{2} + \frac{27}{32} + \frac{243}{512} + \frac{2187}{8192} + \cdots$

   This series is $\frac{128}{27}\sum_{n=0}^{\infty} \left(\frac{9}{16}\right)^n$, which is a convergent geometric series, as the common ratio $\frac{9}{16} < 1$. Thus, the sum of the series is $\frac{128/27}{1-(9/16)} = \frac{128}{27}\frac{16}{7} = \frac{2048}{189} = 10.\overline{835978}$.

5. $\frac{3}{2} + \frac{9}{16} + \frac{27}{72} + \frac{81}{256} + \frac{243}{800} + \frac{729}{2304} + \cdots$

   This series is $\sum_{n=1}^{\infty} \frac{3^n}{n^2 2^n}$. By the Ratio Test, $\lim_{n\to\infty} \left| \frac{3^{n+1}/[(n+1)^2 2^{n+1}]}{3^n/(n^2 2^n)} \right| = \lim_{n\to\infty} \frac{3n^2}{2(n+1)^2} = \frac{3}{2} > 1$, so the series diverges.

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§7.5 #79 Evaluate $\int x\sin^2 x\cos x\,\mathrm{d}x$.

Letting $u = x$ gives $\mathrm{d}v = \sin^2\cos x\,\mathrm{d}x$ and $\mathrm{d}u = \mathrm{d}x$. To determine v, let $w = \sin x$, thus, $\mathrm{d}w = \cos x\,\mathrm{d}x$ and $\int \sin^2 x\cos x\,\mathrm{d}x = \int w^2\,\mathrm{d}w = \frac{w^3}{3} + C = \frac{1}{3}\sin^3 x + C$, thus

$$\int x\sin^2 x\cos x\,\mathrm{d}x = \frac{x}{3}\sin^3 x - \frac{1}{3}\int \sin^3 x\,\mathrm{d}x = \frac{x}{3}\sin^3 x - \frac{1}{3}\int \sin x\,\mathrm{d}x + \frac{1}{3}\int \cos^2 x\sin x\,\mathrm{d}x.$$

Letting $z = \cos x$ gives $\mathrm{d}z = -\sin x\,\mathrm{d}x$ and

$$\int x\sin^2 x\cos x\,\mathrm{d}x = \frac{x}{3}\sin^3 x + \frac{1}{3}\cos x - \frac{1}{3}\int z^2\,\mathrm{d}z = \frac{x}{3}\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x + C.$$

§11.3 #37 (a) Use the sum of the first 10 terms to estimate the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. How good is this estimate?

Let $f(x) = \frac{1}{x^2}$, which is positive when $x > 0$, so $f'(x) = -\frac{2}{x^3} < 0$ for $x > 0$: the integral test can show that the p-series is convergent. The partial sum $\sum_{n=1}^{10} f(n) \approx 1.54976773$. Compared to the value in part (c), this is within $\left|\frac{\pi^2}{6} - \sum_{n=1}^{10} f(n)\right| \approx 0.095$.

(b) Improve this estimate using $s_n + \int_{n+1}^{\infty}\frac{\mathrm{d}x}{x^2} \le \sum_{n=1}^{\infty}\frac{1}{n^2} \le s_n + \int_{n}^{\infty}\frac{\mathrm{d}x}{x^2}$ with $n = 10$.

The integral test provides an interval for the sum of the series, given the first 10 terms: $\int_{11}^{\infty} f(x)\,\mathrm{d}x + \sum_{n=1}^{10} f(n) \le \sum_{n=1}^{\infty} f(n) \le \int_{10}^{\infty} f(x)\,\mathrm{d}x + \sum_{n=1}^{10} f(n)$. Evaluating the integrals: $\int_{x}^{\infty} f(t)\,\mathrm{d}t = \lim_{u\to\infty}\left[-\frac{1}{t}\right]_x^u = \frac{1}{x}$. Thus, $1.640676822 \le \sum_{n=1}^{\infty} f(n) \le 1.64976774$; here the midpoint of the interval $s_n + \frac{1}{2}\left[\frac{1}{10} + \frac{1}{11}\right] \approx 1.645222276621$ is the approximation, and is accurate to two decimal places, or within $\frac{1}{2}\left[\frac{1}{10} - \frac{1}{11}\right] = \frac{1}{220} = 0.00\overline{45}$ of the sum.

(c) Compare your estimate in part (b) with the exact value $\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

$$\left|\frac{\pi^2}{6} - \left(\sum_{n=1}^{10} f(n) + \frac{1}{2}\left[\frac{1}{10} + \frac{1}{11}\right]\right)\right| \approx 0.0002882.$$

(d) Find a value of n that will ensure that the error in the approximation $s \approx s_n$ is less than 0.001.

From above, the value $n \in \mathbb{N}$ must be determined such that $\frac{1}{n} < \frac{1}{1000}$, which gives $n > 1000$. Note that, if the midpoint of the interval is used, the approximation $s_n + \frac{1}{2}\left[\frac{1}{n} + \frac{1}{n+1}\right]$ has an error less than 0.001 when $n > 22$.

§11.4 #37 The meaning of the decimal representation of a number $0.d_1d_2d_3\ldots$ (where the digit d_i is one of the numbers $0, 1, 2, \ldots, 9$) is that $0.d_1d_2d_3d_4\cdots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \cdots$. Show that this series always converges.

The series $\sum_{n=1}^{\infty}\frac{d_n}{10^n}$ can be compared to $\sum_{m=0}^{\infty}\frac{1}{10^m} = \sum_{n=1}^{\infty}\frac{1}{10^{n-1}}$: since $\frac{d_n}{10^n} < \frac{10}{10^n} = \frac{1}{10^{n-1}}$, for $n \in \mathbb{N}$, and the latter series is geometric series with common ratio $r = \frac{1}{10}$ that converges to $\frac{10}{9} = 1.\bar{1}$, the former series always converges.

§11.5 #27 Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$ correct to four decimal places.

This is a convergent alternating series, as $\frac{1}{(2n+2)!} < \frac{1}{(2n)!}$ and $\lim_{n\to\infty} \frac{1}{(2n)!} = 0$. To find a partial sum $s_n = \sum_{k=1}^{n} \frac{(-1)^k}{(2k)!}$ correct to four decimal places, the error estimate $\frac{1}{(2n+2)!} \leq \frac{1}{20000}$ must be satisfied: this gives $(2n+2)! > 20000$, which is true for $2n+2 \geq 8$ or $n \geq 3$, so $s_3 = \sum_{n=1}^{3} \frac{(-1)^n}{(2n)!} = -\frac{331}{720} = -0.4597\bar{2}$ approximates $\cos 1 - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \approx -0.459697694$ correct to four decimal places (when rounded).

§11.6 #47 (a) Find the partial sum s_5 of the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$. Use Exercise 46 to estimate the error in using s_5 as an approximation to the sum of the series.

As indicated, the series converges, as verified by the Ratio Test: $r_n = \frac{a_{n+1}}{a_n} = \frac{1/[(n+1)2^{n+1}]}{1/(n2^n)} = \frac{n}{2n+2} \to \frac{1}{2} < 1$ as $n \to \infty$, with $\{r_n\}$ increasing; thus $R_n \leq \frac{1/[(n+1)2^{n+1}]}{1-(1/2)} = \frac{1}{(n+1)2^n}$, so $s_5 = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \frac{1}{160} = \frac{661}{960} = 0.688541\bar{6}$ approximates $\ln 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n} \approx 0.69314718$ with an error of at most $R_n \leq \frac{1}{192} = 0.005208\bar{3}$.

(b) Find a value of n so that s_n is within 0.00005 of the sum. Use this value of n to approximate the sum of the series.

Since $R_n = s - s_n \leq \frac{1}{(n+1)2^n} \leq \frac{1}{20000}$, gives $20000 \leq (n+1)2^n$, which is true for $n \geq 11$, thus $s_{11} = s_5 + \frac{1}{384} + \frac{1}{896} + \frac{1}{2048} + \frac{1}{4608} + \frac{1}{10240} + \frac{1}{22528} = \frac{4918525}{7096320} \approx 0.6931$.

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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§7.5 #68 Evaluate $\displaystyle\int \frac{x^2}{x^6+3x^3+2}\,\mathrm{d}x$.

Using $u = x^3$ gives $\mathrm{d}u = 3x^2\,\mathrm{d}x$ and

$$\int \frac{x^2}{x^6+3x^3+2}\,\mathrm{d}x = \frac{1}{3}\int \frac{\mathrm{d}u}{u^2+3u+2} = \frac{1}{3}\int \frac{\mathrm{d}u}{u+1} - \frac{1}{3}\int \frac{\mathrm{d}u}{u+2} = \frac{1}{3}\ln\left|\frac{u+1}{u+2}\right| + C = \ln\left|\sqrt[3]{\frac{x^3+1}{x^3+2}}\right| + C.$$

§11.8 #26 Find the radius of convergence and interval of convergence of $\displaystyle\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$.

By the Ratio Test, the power series converges when

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}/(n+1)[\ln(n+1)]^2}{x^{2n}/n(\ln n)^2}\right| = x^2\left(\lim_{n\to\infty}\frac{n}{n+1}\right)\left[\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}\right]^2 = x^2\left[\lim_{n\to\infty}\frac{1/n}{1/(n+1)}\right]^2 = x^2 < 1,$$

thus the radius of convergence is $R = 1$. When $x = \pm 1$, the series is $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$: letting $u = \ln x, \mathrm{d}u = \dfrac{\mathrm{d}x}{x}$ and $\displaystyle\int_2^{\infty}\frac{\mathrm{d}x}{x(\ln x)^2} = \int_{\ln 2}^{\infty}\frac{\mathrm{d}u}{u^2} = \lim_{t\to\infty}\left[-\frac{1}{u}\right]_{\ln 2}^{t} = \frac{1}{\ln 2}$, so the series is convergent. Thus, the interval of convergence of the power series is $-1 \le x \le 1$.

§11.8 #38 If $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n$, where $c_{n+4} = c_n$ for all $n \ge 0$, find the interval of convergence of the series and a formula for $f(x)$.

From the given, $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n = (c_0 + c_1 x + c_2 x^2 + c_3 x^3)\sum_{n=0}^{\infty} x^{4n} = \frac{c_0 + c_1 x + c_2 x^2 + c_3 x^3}{1-x^4} = \frac{c_0+c_1+c_2+c_3}{4(1-x)} + \frac{c_0-c_1+c_2-c_3}{4(1+x)} + \frac{(c_1-c_3)x + c_0 - c_2}{2(1+x^2)}$, which converges whenever $x^4 < 1$, or when $|x| < 1$—this converges at $x = 1$ if $c_0 + c_1 + c_2 + c_3 = 0$ and converges at $x = -1$ if $c_0 + c_2 = c_1 + c_3$.

§11.9 #36 The Bessel function of order 1 is defined by $J_1(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$.

(a) Show that J_1 satisfies the differential equation $x^2 J_1''(x) + xJ_1'(x) + (x^2-1)J_1(x) = 0$.

Noting $J_1'(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)x^{2n}}{n!(n+1)!2^{2n+1}}$ and $J_1''(x) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)x^{2n-1}}{(n-1)!(n+1)!2^{2n}}$,

$$\begin{aligned} x^2 J_1''(x) + xJ_1'(x) + (x^2-1)J_1(x) &= \frac{x}{2} - \frac{x}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n[(2n+1)^2-1]x^{2n+1}}{n!(n+1)!2^{2n+1}} + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{n!(n+1)!2^{2n+1}} \\ &= \sum_{n=1}^{\infty}\frac{(-1)^n 4(n^2+n)x^{2n+1}}{n!(n+1)!2^{2n+1}} - \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{(n-1)!n!2^{2n-1}} = 0. \end{aligned}$$

(b) Show that $J_0'(x) = -J_1(x)$, where J_0 is the Bessel function of order 0 given in Example 4.

Since $J_0(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(n!)^2 2^{2n}}$, $J_0'(x) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{n!(n-1)!2^{2n-1}} = -\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(n+1)!n!2^{2n+1}} = -J_1(x)$.

§11.10 #60 Use series to approximate $\displaystyle\int_0^{0.5} x^2 e^{-x^2}\,\mathrm{d}x$ to within $|\text{error}| < 0.001$.

The Maclaurin series for the integrand is $x^2 e^{-x^2} = x^2\displaystyle\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n!}$, thus

$$\int_0^x t^2 e^{-t^2}\,\mathrm{d}t = \int_0^x \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+2}}{n!}\,\mathrm{d}t = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^x t^{2n+2}\,\mathrm{d}t = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+3)}\left[t^{2n+3}\right]_0^x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{n!(2n+3)}.$$

Thus $\displaystyle\int_0^{0.5} x^2 e^{-x^2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)2^{2n+3}}$, a convergent alternating series, and the error of the partial sum given by $\displaystyle\left|\int_0^{0.5} x^2 e^{-x^2}\,\mathrm{d}x - \sum_{n=0}^{N} \frac{(-1)^n}{n!(2n+3)2^{2n+3}}\right| < \frac{1}{32(N+1)!(2N+5)4^N}$. Setting the error less than 0.001 gives $31.25 < (N+1)!(2N+5)4^N$, which holds when $N \geq 1$, thus $\displaystyle\int_0^{0.5} x^2 e^{-x^2}\,\mathrm{d}x \approx \frac{1}{1\cdot 3\cdot 8} - \frac{1}{1\cdot 5\cdot 32} = \frac{17}{480} = 0.03541\bar{6}$.

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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§7.5 #67 Evaluate $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x+1}+\sqrt{x}}$.

Rationalizing the denominator leads to

$$\int \frac{\mathrm{d}x}{\sqrt{x+1}+\sqrt{x}}\frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}} = \int \frac{(\sqrt{x+1}-\sqrt{x})\,\mathrm{d}x}{(x+1)-x} = \frac{2}{3}[\sqrt{(x+1)^3}-\sqrt{x^3}]+C.$$

§11.8 #37 A function f is defined by $f(x) = 1+2x+x^2+2x^3+x^4+\cdots$, that is, its coefficients are $c_{2n} = 1$ and $c_{2n+1} = 2$ for all $n \geq 0$. Find the interval of convergence of the series and find an explicit formula for $f(x)$.

$f(x) = (1+x+x^2+\cdots)+(x+x^3+x^5+\cdots) = \sum_{n=0}^{\infty} x^n + x\sum_{n=0}^{\infty} x^{2n} = \dfrac{1}{1-x}+\dfrac{x}{1-x^2} = \dfrac{1+2x}{1-x^2}$.

This series converges when $-1 < x < 1$.

§11.9 #41 Use the power series for $\tan^{-1} x$ to prove the following expression for π as the sum of an infinite series: $\pi = 2\sqrt{3}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)3^n}$.

The power series representation $\tan^{-1} x = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+1}}{2n+1}$ holds for $-1 \leq x \leq 1$. It is indicated that $x = \dfrac{1}{\sqrt{3}}$: thus $\tan^{-1}\dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6} = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{\sqrt{3}(2n+1)3^n}$, which gives the series representation of π.

§11.10 #21 Find the Taylor series for $f(x) = \ln x$ centered at $a = 2$. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

By definition of the Taylor series, as $f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{x^n}$,

$$f(x) = \ln x = \sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n = \ln 2 - \sum_{n=1}^{\infty}\frac{(-1)^n(n-1)!}{2^n n!}(x-2)^n = \ln 2 - \sum_{n=1}^{\infty}\left(\frac{-1}{2}\right)^n\frac{(x-2)^n}{n}.$$

By the ratio test, the Taylor series converges when

$$\lim_{n\to\infty}\left|\frac{(-1/2)^{n+1}[(x-2)^{n+1}/(n+1)]}{(-1/2)^n[(x-2)^n/n]}\right| = \frac{|x-2|}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x-2|}{2} < 1 \Longrightarrow |x-2| < 2,$$

which gives $0 < x < 4$. When $x = 0$, the series becomes $\ln 2 - \sum_{n=1}^{\infty}\dfrac{1}{n}$, which is divergent (which makes sense, as $\ln x$ is undefined at $x = 0$); when $x = 4$, the series becomes $\ln 2 - \sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}$, which is converges to $2\ln 2 = \ln 4$. Thus, the Taylor series for $\ln x$ centered at $a = 2$ has an interval of convergence $0 < x \leq 4$.

§11.10 #51 (a) Use the binomial series (if k is any real number and $|x| < 1$),

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots,$$

where $\dbinom{k}{n} = \dfrac{k(k-1)\cdots(k-n+1)}{1\cdot 2\cdot\cdots\cdot n}$, to expand $\dfrac{1}{\sqrt{1-x^2}}$.

The binomial series representation, which converges on $-1 < x < 1$, is

$$(1-x^2)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-x^2)^n = 1+\frac{x^2}{2}+\cdots+\frac{[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)]x^{2n}}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}+\cdots = \sum_{n=0}^{\infty}\frac{(2n)!x^{2n}}{4^n(n!)^2}.$$

(b) Use part (a) to find the Maclaurin series for $\sin^{-1} x$.

The power series representation, with radius of convergence 1, is

$$\sin^{-1} x = \int_0^x \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} \int_0^x t^{2n}\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(2n)!x^{2n+1}}{4^n (n!)^2 (2n+1)}.$$

The Root Test, using Stirling's approximation of $n!$, shows that the series representing $\sin^{-1}(\pm 1)$ converges to $\pm\pi/2$.

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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§7.5 #38 Evaluate $\displaystyle\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,\mathrm{d}x$.

Multiplying by $\dfrac{\sin x}{\sin x}$ and using $u = 4\sin x - \cos x$ gives

$$\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,\mathrm{d}x = \int_{\pi/4}^{\pi/2} \frac{\sin x + 4\cos x}{4\sin x - \cos x}\,\mathrm{d}x = \int_{3/\sqrt{2}}^{4} \frac{\mathrm{d}u}{u} = [\ln|u|]_{3/\sqrt{2}}^{4} = \frac{5}{2}\ln 2 - \ln 3 \approx 0.63425566.$$

§7.7 #44 Use the Trapezoidal Rule with $n = 10$ to approximate $\displaystyle\int_0^{20} \cos(\pi x)\,\mathrm{d}x$. Compare your result to the actual value. Can you explain the discrepancy?

Since $\cos(2k\pi) = 1 = \max\cos(\pi x)$, $k \in \int$, the Trapezoidal Rule with $n = 1, 2, 5, 10$ will give an approximation of $\displaystyle\int_0^{20} \mathrm{d}x = 20$ for $\displaystyle\int_0^{20} \cos(\pi x)\,\mathrm{d}x = \left[\frac{\sin(\pi x)}{\pi}\right]_0^{20} = 0$—that is, it approximates the function $y = \cos(\pi x)$ as $y = 1$.

§7.8 #72 Estimate the numerical value of $\displaystyle\int_0^{\infty} e^{-x^2}\,\mathrm{d}x$ by writing it as the sum of $\displaystyle\int_0^4 e^{-x^2}\,\mathrm{d}x$ and $\displaystyle\int_4^{\infty} e^{-x^2}\,\mathrm{d}x$. Approximate the first integral by using Simpson's Rule with $n = 8$ and show that the second integral is smaller than $\displaystyle\int_4^{\infty} e^{-4x}\,\mathrm{d}x$, which is less than 0.0000001.

As indicated, $\displaystyle\int_0^{\infty} e^{-x^2}\,\mathrm{d}x = \int_0^4 e^{-x^2}\,\mathrm{d}x + \int_4^{\infty} e^{-x^2}\,\mathrm{d}x$. Using Simpson's (1/3) Rule, with $n = 8$,

$$\int_0^4 e^{-x^2}\,\mathrm{d}x = \frac{1}{6}\left[1 + \frac{4}{\sqrt[4]{e}} + \frac{2}{e} + \frac{4}{\sqrt[4]{e^9}} + \frac{2}{e^4} + \frac{4}{\sqrt[4]{e^{25}}} + \frac{2}{e^9} + \frac{4}{\sqrt[4]{e^{49}}} + \frac{1}{e^{16}}\right] \approx 0.8861963466352.$$

(Using power series,

$$\int_0^4 e^{-x^2}\,\mathrm{d}x = \int_0^4 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,\mathrm{d}x = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+1}}{n!(2n+1)}\right]_0^4 = 4\sum_{n=0}^{\infty} \frac{(-16)^n}{n!(2n+1)},$$

whose partial sums through $n = N$ have an error bounded by $\dfrac{4\cdot 16^{N+1}}{(N+1)!(2N+3)} \le \dfrac{1}{2\cdot 10^8}$, which gives $128\cdot 10^8 \le \dfrac{(N+1)!(2N+3)}{16^N}$, which is true when $N \ge 54$, which approximates the integral as 0.8862269155.)

For $x \ge 4$, since e^x is an increasing function, $0 < e^{4x} \le e^{x^2}$ and $0 < e^{-x^2} \le e^{-4x}$, thus

$$\int_4^{\infty} e^{-x^2}\,\mathrm{d}x \le \int_4^{\infty} e^{-4x}\,\mathrm{d}x = -\frac{1}{4}\lim_{t\to\infty}\int_4^t -4e^{-4x}\,\mathrm{d}x = \frac{1}{4}\left[\frac{1}{e^{16}} - \lim_{t\to\infty}\frac{1}{e^{4t}}\right] = \frac{1}{4e^{16}} \approx 2.8\times 10^{-8}.$$

Thus, by the indicated approach, $\displaystyle\int_0^{\infty} e^{-x^2}\,\mathrm{d}x \approx 0.886196374769$.

(This only correctly rounds to the exact value up to four decimal places. If $c = 4$ is replaced by $c = 3$, the approximation would be 0.8861725695 with the decaying error term bounded by 4.1×10^{-5}, which still correctly rounds to the exact value up to four decimal places.)

§11.11 #26 How many terms of the Maclaurin series for $\ln(1+x)$ do you need to use to estimate $\ln 1.4$ to within 0.001?

From the Maclaurin series, $\ln 1.4 = \ln(1+0.4) = -\sum_{n=1}^{\infty} \dfrac{(-1)^n 2^n}{n5^n}$, which is an alternating series, whose error is bounded as $\left| -\sum_{n=N+1}^{\infty} \frac{(-1)^n 2^n}{n5^n} \right| < \frac{2^{N+1}}{(N+1)5^{N+1}} \leq \frac{1}{1000}$, which gives $400 \cdot 2^N \leq (N+1)5^N$, which is true when $N \geq 5$, so five terms provide the required accuracy:

$$\ln 1.4 \approx 0.33647 \approx -\sum_{n=1}^{5} \frac{(-1)^n 2^n}{n5^n} = \frac{2}{5} - \frac{2}{25} + \frac{8}{375} - \frac{4}{625} + \frac{32}{15625} = \frac{15796}{46875} = 0.336981\bar{3}.$$

§11.11 #30 Suppose you know that $f^{(n)}(4) = \dfrac{(-1)^n n!}{3^n(n+1)}$ and the Taylor series of f centered at 4 converges to $f(x)$ for all x in the interval of convergence. Show that the fifth-degree Taylor polynomial approximates $f(5)$ with error less than 0.0002.

The Taylor series of f centered at 4 is $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)}(x-4)^n$, which has a radius of convergence determined as follows:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-4)^{n+1}/[3^{n+1}(n+2)]}{(-1)^n(x-4)^n/[3^n(n+1)]} \right| = \frac{|x-4|}{3} \lim_{n\to\infty} \frac{n+1}{n+2} = \frac{|x-4|}{3} < 1 \Longrightarrow 1 < x < 7.$$

So the series at $x = 5$ converges to $f(5)$: since the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)}$ is an alternating series, the truncation error can be bounded as follows:

$$|f(5) - T_5(5)| = \left| \sum_{n=6}^{\infty} \frac{(-1)^n}{3^n(n+1)} \right| < \frac{1}{7 \cdot 3^6} = \frac{1}{5103} \approx 0.000195963.$$

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition

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This is an open-book quiz. Work quietly and individually. You may use a calculator, but not any internet-enabled devices. Make sure that there is a seat between you and your neighbors.

1. Evaluate $\operatorname{erf}(1)$ to within 0.00005, where $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\,\mathrm{d}t$. **[3 pts]**

   Using the Maclaurin series for $\exp(x)$,

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \qquad \Longrightarrow \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\,\mathrm{d}t = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}$$

$$\Longrightarrow \operatorname{erf}(1) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}.$$

   Using the alternating series error estimate,

$$\left|\operatorname{erf}(1) - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)n!}\right| < \frac{2}{\sqrt{\pi}(2N+3)(N+1)!} < \frac{1}{20000} \Longrightarrow \frac{40000}{\sqrt{\pi}} < 23000 < (N+1)!(2N+3),$$

   which is true for $N \geq 6$, thus,

$$\operatorname{erf}(1) \approx 0.8427 \approx \frac{2}{\sqrt{\pi}} \sum_{n=0}^{6} \frac{(-1)^n}{(2n+1)n!} = \frac{2}{\sqrt{\pi}}\left(1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} + \frac{1}{9360}\right) = \frac{1614779}{1081080\sqrt{\pi}},$$

   or about 0.842714222381.

2. Evaluate $S(1)$ to within 0.00005, where $S(x) = \displaystyle\int_0^x \sin\left(\frac{\pi t^2}{2}\right)\mathrm{d}t$. **[3 pts]**

   Using the Maclaurin series for $\sin(x)$,

$$\sin\left(\frac{\pi t^2}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi t^2)^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{-\pi^2}{4}\right)^n \frac{t^{4n+2}}{(2n+1)!}$$

$$\Longrightarrow S(x) = \int_0^x \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{-\pi^2}{4}\right)^n \frac{t^{4n+2}}{(2n+1)!}\,\mathrm{d}t = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{-\pi^2}{4}\right)^n \frac{x^{4n+3}}{(2n+1)!(4n+3)}$$

$$\Longrightarrow S(1) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{-\pi^2}{4}\right)^n \frac{1}{(2n+1)!(4n+3)}.$$

   Using the alternating series error estimate,

$$\left|S(1) - \frac{\pi}{2} \sum_{n=0}^{N} \left(\frac{-\pi^2}{4}\right)^n \frac{1}{(2n+1)!(4n+3)}\right| < \frac{\pi}{2}\, \frac{\pi^{2N+2}}{4^{N+1}(4N+7)(2N+3)!} < \frac{1}{20000}$$

$$\Longrightarrow 2500\pi^3 < 78000 < \left(\frac{2}{5}\right)^N (4N+7)(2N+3)! < \left(\frac{4}{\pi^2}\right)^N (4N+7)(2N+3)!,$$

   which is true for $N \geq 3$, thus,

$$S(1) \approx 0.43825914739 \approx \frac{\pi}{2} \sum_{n=0}^{3} \left(\frac{-\pi^2}{4}\right)^n \frac{1}{(2n+1)!(4n+3)} = \frac{\pi}{2}\left(\frac{1}{3} - \frac{\pi^2}{168} + \frac{\pi^4}{21120} - \frac{\pi^6}{4838400}\right)$$

$$\approx \frac{(17740800 - 316800\pi^2 + 2520\pi^4 - 11\pi^6)\pi}{106444800} \approx 0.4382508575.$$

3. Using ten trapezoids, approximate the area of the region bounded by the curve $y = \sin\left(\dfrac{\pi x^2}{2}\right)$, the line $x = 1$ and the coordinate axes. Estimate the error of the approximation. **[2 pts]**

The area is given by $S(1)$ and approximated by the Trapezoid rule, with $n = 10$, as

$$S(1) \approx \frac{1}{10}\sum_{k=1}^{9}\sin\left(\frac{\pi k^2}{200}\right) + \frac{1}{20} = \frac{1}{10}\left[\sin\left(\frac{\pi}{200}\right) + \sin\left(\frac{\pi}{50}\right) + \sin\left(\frac{9\pi}{200}\right) + \sin\left(\frac{2\pi}{25}\right) + \sin\left(\frac{\pi}{8}\right) + \sin\left(\frac{9\pi}{50}\right)\right.$$
$$\left. + \sin\left(\frac{49\pi}{200}\right) + \sin\left(\frac{8\pi}{25}\right) + \sin\left(\frac{81\pi}{200}\right) + \frac{1}{2}\right] \approx 0.438263292.$$

Since

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\sin\left(\frac{\pi x^2}{2}\right)\right] = \pi\frac{\mathrm{d}}{\mathrm{d}x}\left[x\cos\left(\frac{\pi x^2}{2}\right)\right] = \pi\left[\cos\left(\frac{\pi x^2}{2}\right) - \pi x^2\sin\left(\frac{\pi x^2}{2}\right)\right],$$

which is maximized as $M = \pi$ when $x = 0$ over the interval $[0, 1]$, by the error estimate,

$$\left|S(1) - \frac{1}{10}\sum_{k=1}^{9}\sin\left(\frac{\pi k^2}{200}\right) + \frac{1}{20}\right| < \frac{M(1-0)^3}{12(10)^2} = \frac{\pi}{1200} \approx 0.002618.$$

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§7.5 #81 Evaluate $\int \sqrt{1-\sin x}\,\mathrm{d}x$.

Multiplying the integrand by $\sqrt{\dfrac{1+\sin x}{1+\sin x}}$, using $u = \sin x$,

$$\int \sqrt{1-\sin x}\,\mathrm{d}x = \int \frac{\sqrt{1-\sin^2 x}\,\mathrm{d}x}{\sqrt{1+\sin x}} = \int \frac{\cos x\,\mathrm{d}x}{\sqrt{1+\sin x}} = \int \frac{\mathrm{d}u}{\sqrt{1+u}} = 2\sqrt{1+u}+C = 2\sqrt{1+\sin x}+C.$$

§7.7 #15 Use (a) the Trapezoidal Rule, and (c) Simpson's Rule to approximate $\displaystyle\int_0^1 \frac{x^2}{1+x^4}\,\mathrm{d}x$ with $n = 10$.

By the Trapezoidal Rule, with $n = 10$,

$$\int_0^1 \frac{x^2}{1+x^4}\,\mathrm{d}x \approx \frac{1}{10}\left[\sum_{k=1}^{9} \frac{(k/10)^2}{1+(k/10)^4} + \frac{1}{2}\left(0+\frac{1}{2}\right)\right] = \frac{1}{40} + \sum_{k=1}^{9} \frac{10k^2}{10000+k^4} \approx 0.2437469.$$

By Simpson's (1/3) Rule, with $n = 10$,

$$\begin{aligned}\int_0^1 \frac{x^2}{1+x^4}\,\mathrm{d}x &\approx \frac{1}{30}\left[2\sum_{k=1}^{9} \frac{(k/10)^2}{1+(k/10)^4} + 2\sum_{k=1}^{5} \frac{[(2k-1)/10]^2}{1+[(2k-1)/10]^4} + \frac{1}{2}\right] \\ &\approx \frac{20}{3}\sum_{k=1}^{9} \frac{k^2}{10000+k^4} + \frac{20}{3}\sum_{k=1}^{5} \frac{(2k-1)^2}{10000+(2k-1)^4} + \frac{1}{60} \approx 0.24375128921.\end{aligned}$$

Noting $x^4+1 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$ and $x^2 \pm \sqrt{2}x + 1 = \left(x \pm \frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2$, it can be shown that

$$\begin{aligned}\int_0^1 \frac{x^2}{1+x^4}\,\mathrm{d}x &= \left[\frac{\sqrt{2}}{8}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + \frac{\sqrt{2}}{4}[\tan^{-1}(\sqrt{2}x+1)+\tan^{-1}(\sqrt{2}x-1)]\right]_0^1 \\ &= \frac{\sqrt{2}}{8}[\ln(3-2\sqrt{2})+\pi] \approx 0.243747747.\end{aligned}$$

§7.8 #55 The integral $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,\mathrm{d}x$ is improper for two reasons: The interval $[0,\infty)$ is infinite and the integrand has an infinite discontinuity at 0. Evaluate it by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows:

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,\mathrm{d}x = \int_0^1 \frac{1}{\sqrt{x}(1+x)}\,\mathrm{d}x + \int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,\mathrm{d}x.$$

Using $u = \sqrt{x}$ gives $\mathrm{d}u = \dfrac{\mathrm{d}x}{2\sqrt{x}}$, $1+x = 1+u^2$ and

$$\int_a^b \frac{1}{\sqrt{x}(1+x)}\,\mathrm{d}x = \int_{\sqrt{a}}^{\sqrt{b}} \frac{2\,\mathrm{d}u}{u^2+1} = \left[2\tan^{-1}u\right]_{\sqrt{a}}^{\sqrt{b}} = 2\tan^{-1}\sqrt{b} - 2\tan^{-1}\sqrt{a},$$ thus, from above,

$$\begin{aligned}\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,\mathrm{d}x &= \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}(1+x)}\,\mathrm{d}x + \lim_{u\to\infty}\int_1^u \frac{1}{\sqrt{x}(1+x)}\,\mathrm{d}x \\ &= 2\tan^{-1}1 - 2\lim_{t\to 0^+}\tan^{-1}\sqrt{u} + 2\lim_{u\to\infty}\tan^{-1}\sqrt{u} - 2\tan^{-1}1 = \pi.\end{aligned}$$

§7.8 #73 If $f(t)$ is continuous for $t \geq 0$, the *Laplace transform* of f is the function F defined by

$$F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$$

and the domain of F is the set consisting of all numbers for which the integral converges. Find the Laplace transforms for the following functions.

(a) $f(t) \equiv 1$

The Laplace transform of the function 1 is given by

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\,\mathrm{d}t = \lim_{u\to\infty}\int_0^u e^{-st}\,\mathrm{d}t = \lim_{u\to\infty}\left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=u} = \frac{1}{s}\lim_{u\to\infty}\left(1-\frac{1}{e^{su}}\right) = \frac{1}{s},\text{ when } s>0.$$

Note that the transform is divergent when $s \leq 0$.

(b) $f(t) = e^t$

The Laplace transform of the function e^t is given by

$$\mathcal{L}\left[e^t\right] = \int_0^\infty e^t e^{-st}\,\mathrm{d}t = \lim_{u\to\infty}\int_0^u e^{(1-s)t}\,\mathrm{d}t = \lim_{u\to\infty}\left[\frac{e^{(1-s)t}}{1-s}\right]_{t=0}^{t=u} = \frac{1}{s-1}\lim_{u\to\infty}\left(1-\frac{1}{e^{(s-1)u}}\right)$$
$$= \frac{1}{s-1},\text{ when } s>1.$$

Note that the transform is divergent when $s \leq 1$.

In general, $\mathcal{L}[e^{at}] = \dfrac{1}{s-a}$, $s>a$, shifting $\mathcal{L}[1]$ a units to the right, i.e. $\mathcal{L}[e^{at}](s) = \mathcal{L}[1](s-a)$.

(c) $f(t) = t$

Letting $v = t$ and $\mathrm{d}w = e^{-st}\,\mathrm{d}t$ gives $w = -\dfrac{e^{-st}}{s}$ and $\mathrm{d}v = \mathrm{d}t$, so the Laplace transform of the function t is given by

$$\mathcal{L}[t] = \int_0^\infty te^{-st}\,\mathrm{d}t = \lim_{u\to\infty}\int_0^u te^{-st}\,\mathrm{d}t = \lim_{u\to\infty}\left(\left[-\frac{te^{-st}}{s}\right]_{t=0}^{t=u} + \frac{1}{s}\int_0^u e^{-st}\,\mathrm{d}t\right)$$
$$= -\lim_{u\to\infty}\left(\left[\frac{te^{-st}}{s}\right]_{t=0}^{t=u} + \left[\frac{e^{-st}}{s^2}\right]_{t=0}^{t=u}\right) = \frac{1}{s^2}\lim_{u\to\infty}\left(1-\frac{su+1}{e^{su}}\right) = \frac{1}{s^2}\left(1-\lim_{u\to\infty}\frac{s}{se^{su}}\right) = \frac{1}{s^2},$$

with one implementation of L'Hôpital's rule, when $s>0$. Note that the transform is divergent when $s \leq 0$.

In can be shown that, in general, $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$, $s>0$, for positive integer n.

§11.11 #39 In §4.8 we considered Newton's method for approximating a root r of the equation $f(x) = 0$, and from an initial approximation x_1 we obtained successive approximations $x_2, x_3, \ldots$, where $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. Use Taylor's Inequality with $n = 1$, $a = x_n$, and $x = r$ to show that if $f''(x)$ exists on an interval I containing r, x_n, and x_{n+1}, and $|f''(x)| \leq M$, $|f'(x)| \geq K$ for all $x \in I$, then $|x_{n+1} - r| \leq \dfrac{M}{2K}|x_n - r|^2$.

For $a = x_n$, $f(x) \approx T_1(x) = f(x_n) + f'(x_n)(x - x_n)$: $T_1(x_{n+1}) = 0$ gives $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. By Taylor's Inequality, $|f(r) - T_1(r)| = |f(r) - T_1(x_{n+1}) + T_1(x_{n+1}) - T_1(r)| = |T_1(x_{n+1}) - T_1(r)| = |f'(x_n)(x_{n+1} - r)| < \dfrac{M}{2}|r - x_n|^2$, which gives $|x_{n+1} - r| < \dfrac{M}{2|f'(x_n)|}|x_n - r|^2 \leq \dfrac{M}{2K}|x_n - r|^2$.

Source: James Stewart, *Calculus Early Transcendentals*, 8e, International Metric Edition